FRAMES IN FINITE-DIMENSIONAL INNER PRODUCT SPACES

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1
Introduction

One of the important concepts in the study of vector spaces is the concepts of a basis for the vectors spaces, which allows every vector to be uniquely represented as a linear combination of the basis elements. However, the linear independence property for a basis is restrictive; sometimes it is impossible to find vector which both fulfill the basis requirements and also satisfy external condition demanded by applied problems. For such purpose, we need to look for more flexible type of spanning sets.

Frames are such tools which provide these alternatives. They not only have great variety for use in applications, but also have a rich theory from a pure analysis point of view. A frame for a vector space equipped with an inner product also allows each element in the space to be written as a linear combination of the elements in the frame, but linear independence between the frame elements is not required. Intuitively, one can think about a frame as a basis to which one has added more elements. The theory for frames and bases has developed rapidly in recent years because of its role as a mathematical tool in signal and image processing.

Let’s say you want to send a signal across some kind of communication system, perhaps by talking on wireless phone or sending a photo to your friend over the internet. We think that signal as a vector in a vector space. The way it get transmitted is as a sequence of coefficients which represent the signal in term of a spanning set. If that spanning set is an orthonormal basis, then computing those coefficients just involves finding some inner product of vectors, which a computer can accomplish very quickly. As a result, there is not a significant time delay in sending your voice or the photograph. This is a good feature for a communication system to have, so orthonormal bases are used a lot in such situation.
Orthogonality is a very restrictive property, though. What if one of the coefficients representing a vector gets lost in transmission? That piece of information cannot be reconstructed. It is lost. Perhaps, we’d like our system to have some redundancy, so that if one piece gets lost, the information can be pieced together from what does get through. This is where frames come in.

By using a frame instead of an orthonormal basis, we do give up the uniqueness of coefficients and orthogonality of the vectors. However, these properties are superfluous. If you are sending your side of a phone conversation or a photo, what matter is quickly computing a working set of expansion coefficients, not whether those coefficients are unique. In fact, in some setting the linear independence and orthogonality restrictions inhibit the use of orthonormal bases. Frame can be constructed with a wider variety of characteristics, and can thus be tailored to match the needs of a particular system.

This thesis will introduce the concept of a frame for a finite-dimensional Hilbert space. We begin with the characteristics of frames. The first section discusses some basic facts about frames, giving a standard definition of a frame. Then proceed in the latter to understood a litter bit about what frames are?, and how to construct a frames in finite-dimensional spaces. From that, thinking about the connection between frames in finite-dimensional vector spaces and the infinite-dimensional constructions.

Most of contents of the thesis come from the book "An Introduction to Frames and Riesz Bases", written by Prof. Ole Christensen. Moreover, I have used some results of Prof. Ingrid Daubechies in [2] and some results of Prof. Nguyễn Hữu Việt Hưng in [3].
Frames in Finite-dimensional Inner Product Spaces

1 Some basic facts about frames

Let \( V \) be a finite-dimensional vector space, equipped with an inner product \( \langle \cdot, \cdot \rangle \).

**Definition 1.1** [1] A countable family of elements \( \{f_k\}_{k \in I} \) in \( V \) is a frame for \( V \) if there exist constants \( A, B > 0 \) such that

\[
A \|f\|^2 \leq \sum_{k \in I} |\langle f, f_k \rangle|^2 \leq B \|f\|^2, \quad \forall f \in V.
\] (1.1)

- \( A, B \) are called frame bounds, and they are not unique! Indeed, we can choose \( A' = A^2 \), \( B' = B + 1 \) as other frame bounds.

- The frame are normalized if \( \|f_k\| = 1 \), \( \forall k \in I \).

- In the finite-dimensional space, \( \{f_k\}_{k \in I} \) can be having infinitely many elements by adding infinitely many zero elements to the given frame.

Now, we will only consider finite families \( \{f_k\}_{k \in I} \), \( I \) finite. Then, the upper frame condition is automatically satisfied by Cauchy-Schwartz's inequality

\[
\sum_{k=1}^{m} |\langle f, f_k \rangle|^2 \leq \sum_{k=1}^{m} \|f_k\|^2 \|f\|^2, \quad \forall f \in V.
\] (1.2)
1. Some basic facts about frames

For all \( f \in V \), we are easily to prove that

\[
|\langle f, f_k \rangle|^2 \leq \|f_k\|^2 \|f\|^2, \quad \forall k \in \{1, 2, 3, ..., m\}. \tag{1.3}
\]

Indeed, for each \( k \in \{1, 2, ..., m\} ; f, f_k \in V \).

If \( f = 0 \) then

\[
|\langle f, f_k \rangle| = |\langle 0, f_k \rangle| = 0 = \|f\|^2 \|f_k\|^2.
\]

So, the result holds automatically.

Now assume \( f \neq 0 \), for any \( \lambda \in \mathbb{C} \) we have:

\[
0 \leq \|f_k + \lambda f\|^2 = \langle f_k + \lambda f, f_k + \lambda f \rangle. \tag{1.4}
\]

Expanding the right side

\[
0 \leq \langle f_k, f_k \rangle + \overline{\lambda} \langle f_k, f \rangle + \lambda \langle f, f_k \rangle + |\lambda|^2 \langle f, f \rangle
\]

\[
= \|f_k\|^2 + \overline{\lambda} \langle f_k, f \rangle + \lambda \langle f, f_k \rangle + |\lambda|^2 \|f\|^2.
\]

Now select

\[
\lambda = -\frac{\langle f_k, f \rangle}{\|f\|^2}.
\]

Substituting this into preceding expression yields

\[
0 \leq \|f_k\|^2 - 2 \left| \frac{\langle f, f_k \rangle}{\|f\|^2} \right|^2 + \frac{\left| \frac{\langle f, f_k \rangle}{\|f\|^2} \right|^2}{\|f\|^2}
\]

\[
= \|f_k\|^2 - \frac{\left| \frac{\langle f, f_k \rangle}{\|f\|^2} \right|^2}{\|f\|^2}.
\]

which yields

\[
\left| \frac{\langle f, f_k \rangle}{\|f\|^2} \right|^2 \leq \|f\|^2 \|f_k\|^2.
\]

Thus, we can choose the upper frame bound \( B = \sum_{k=1}^{m} \|f_k\|^2 \).

Recall the Lemma about a vector spaces decomposition.

**Definition 1.2** Let \( W \) is a subspace of a finite-dimensional vector space \( V \). Then

\[
W^\perp = \{ \alpha \in V | \alpha \perp W, \text{ i.e., } \langle \alpha, \beta \rangle = 0, \forall \beta \in W \} \tag{1.5}
\]

is said to be orthogonal complement of \( W \) in \( V \).
1. Some basic facts about frames

We have the following lemma:

**Lemma 1.3** [3] Let $W$ be a subspace of finite-dimensional vector space $V$. Then $(W^\perp)^\perp = W$, and $V$ can be decomposed as $V = W \oplus W^\perp$.

**Proof** Let $(e_1, e_2, ..., e_m)$ be an orthogonal basis of $W$, and extend it to be a basis $(e_1, e_2, ..., e_m, \alpha_{m+1}, ..., \alpha_n)$ of $V$. Applying Schmidt’s orthogonalization process to this basis. Then we obtain an orthogonal basis $(e_1, e_2, ..., e_m, e_{m+1}, ..., e_n)$ for $V$. The vectors $e_{m+1}, ..., e_n$ are orthogonal to each element in $(e_1, e_2, ..., e_m)$, then they are orthogonal to $W$. So, $e_{m+1}, ..., e_n \in W^\perp$.

Let $\alpha \in W$ then $\langle \alpha, \beta \rangle = 0$, $\forall \beta \in W^\perp$. So, $\alpha \in (W^\perp)^\perp$. Therefore, $W \subset (W^\perp)^\perp$.

In addition, if $\alpha$ is an arbitrary vector in $(W^\perp)^\perp \subseteq V$ then

$$
\alpha = a_1 e_1 + a_2 e_2 + ... + a_n e_n.
$$

Since $e_{m+1}, ..., e_n \in W^\perp$ then

$$
0 = \langle \alpha, e_j \rangle = a_1 \langle e_1, e_j \rangle + a_2 \langle e_2, e_j \rangle + ... + a_n \langle e_n, e_j \rangle =
= a_j \langle e_j, e_j \rangle \quad \forall j = m+1, n.
$$

Then, $a_{m+1} = a_{m+2} = ... = a_n = 0$.

So, $\alpha$ represents linearly in $(e_1, ..., e_m)$.

Therefore, $\alpha \in W$. Thus, $(W^\perp)^\perp \subseteq W$.

Hence, $(W^\perp)^\perp = W$.

Finally, we will prove that $W \cap W^\perp = \{0\}$.

Indeed, if $\alpha \in W \cap W^\perp$ then $\|\alpha\|^2 = \langle \alpha, \alpha \rangle = 0$. Thus $\alpha = 0$.

In conclusion,

$$
V = \text{span}(e_1, e_2, ..., e_m) \oplus W^\perp \text{span}(e_{m+1}, ..., e_n) = W \oplus W^\perp.
$$

In order for the lower condition to be satisfied, if and only if $\text{span}\{f_k\}_{f=1}^m = V$. Then we have the following theorem:

**Theorem 1.4** [1] A family of elements $\{f_k\}_{k=1}^m$ in $V$ is a frame for $V$ if and only if $\text{span}\{f_k\}_{k=1}^m = V$. 

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1. Some basic facts about frames

PROOF:
If \( \text{span} \{ f_k \}_{k=1}^m = V \) then we consider the following mapping:

\[
\phi : V \rightarrow \mathbb{R}
\]

\[
f \mapsto \sum_{k=1}^m |\langle f, f_k \rangle|^2.
\]

Firstly, we will prove that there exist \( A, B > 0 \) such that

\[
A \| f \|^2 \leq \sum_{k \in I} |\langle f, f_k \rangle|^2 \leq B \| f \|^2, \quad \forall f \in V, \| f \| = 1. \tag{1.6}
\]

Take \( B = m \sum_{k=1}^m |f_k|^2 \) then (1.6) holds automatically by (1.2). So, we only need to show the existence of \( A \).

For any \( f, g \in V \), we have:

\[
|\phi(f) - \phi(g)| = | \sum_{k=1}^m (|\langle f, f_k \rangle|^2 - |\langle g, f_k \rangle|^2) |
\]

\[
= | \sum_{k=1}^m (|\langle f, f_k \rangle| - |\langle g, f_k \rangle|)(|\langle f, f_k \rangle| + |\langle g, f_k \rangle|) |
\]

\[
\leq \sum_{k=1}^m |\langle f - g, f_k \rangle| (\| f \| \| f_k \| + \| g \| \| f_k \| )
\]

\[
\leq \sum_{k=1}^m \| f - g \| (\| f \| + \| g \| ) \| f_k \|^2
\]

\[
= \| f - g \| (\| f \| + \| g \| ) \sum_{k=1}^m \| f_k \|^2.
\]

Then, \( \phi(f) \) tends to \( \phi(g) \) as \( f \) tends to \( g \). So, \( \phi \) is continuous.

Moreover, the set \( \{ f \in V \mid \| f \| = 1 \} \) , the unit sphere in finite-dimensional space \( V \), is compact. By using Weierstrass’s theorem, we know that \( \phi(f) \) has a minimum on the unit sphere.

Choose \( A = \min_{\| f \| = 1} \phi(f) \), we will prove that \( \phi(f) > 0 \) when \( \| f \| = 1 \), then \( A > 0 \).

Suppose the contrary that \( \phi(f) = 0 \) for some \( f \in V, \| f \| = 1 \). We have:

\[
\phi(f) = \sum_{k=1}^m |\langle f, f_k \rangle|^2 = 0.
\]

It implies that

\[
|\langle f, f_k \rangle| = 0, \forall k = \{1, 2, ..., n\}.
\]
Besides, $f \in V = \text{span} \{f_k\}_{k=1}^m$ then $\exists \{a_k\}_{k=1}^m : f = \sum_{k=1}^m a_k f_k$ and

$$1 = \|f\|^2$$

$$= \langle f, f \rangle = \langle f, \sum_{k=1}^m a_k f_k \rangle$$

$$= \sum_{k=1}^m \alpha_k \langle f, f_k \rangle = 0.$$ 

It is impossible!

Thus, $\phi(f) > 0, \forall f \in V$ when $\|f\| = 1$. Then for $A > 0$, we have

$$\sum_{k=1}^m |\langle f, f_k \rangle|^2 \geq A = A\|f\|^2, \forall f \in V, \|f\| = 1.$$ 

So the lower bound could be chosen as $A = \min_{\|f\| = 1} \phi(f) > 0$.

Indeed, if $f = 0$ then

$$0 = \sum_{k=1}^m |\langle f, f_k \rangle|^2 = A\|f\|^2$$

If $f \neq 0$, we can choose $f' = \frac{f}{\|f\|}$ then $\|f'\| = 1$.

Apply the above result then

$$\sum_{k=1}^m |\langle f', f_k \rangle|^2 \geq A$$

equivalent to

$$\sum_{k=1}^m |\langle \frac{f}{\|f\|}, f_k \rangle|^2 \geq A.$$ 

Hence,

$$\sum_{k=1}^m |\langle f, f_k \rangle|^2 \geq A\|f\|^2, \forall f \in V.$$ 

Conversely, we claim that if $\{f_k\}_{k=1}^m$ is a frame for $V$ then $\text{span} \{f_k\}_{k=1}^m = V$. Indeed, we suppose the contrary that $\{f_k\}_{k=1}^m$ is a frame for $V$ but $\text{span} \{f_k\}_{k=1}^m \subsetneq V$.

Denote $W = \text{span} \{f_k\}_{k=1}^m \subsetneq V$. Then there exists $f \in V$ but $f \notin W$, i.e., we can write $f = g + h$ where $g \in W^\perp \setminus \{0\}$ and $h \in W$ (by Lemma 1.3).
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So, \( g \) is orthogonal to \( \text{span}\{f_k\}_{k=1}^m \).
It implies that \( \sum_{k=1}^{m} |\langle g, f_k \rangle|^2 = 0 \). By the definition of frame, \( g = 0 \). Contradiction!
Hence, \( \text{span}\{f_k\}_{k=1}^m = V \).

**Remark 1.5**

- A frame might contain more elements than needed to be a basic of \( V \) since \( \text{span}\{f_k\}_{k=1}^m = V \). In particular, if \( \{f_k\}_{k=1}^m \) is a frame for \( V \) and \( \{g_k\}_{k=1}^n \) is an arbitrary finite collection of vectors in \( V \), then \( \{f_k\}_{k=1}^m \cup \{g_k\}_{k=1}^n \) is also a frame for \( V \). Because \( \text{span}\{f_k\}_{k=1}^m \cup \{g_k\}_{k=1}^n \} = \text{span}\{f_k\}_{k=1}^m = V \).
- A frame which is not a basis is said to be overcomplete or redundant.

**Example 1.6**

\( V = \mathbb{R}^3, \ f_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \ f_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \ f_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \ f_4 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \).

For all \( f \in \mathbb{R}^3, f = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \) for \( a, b, c \in \mathbb{R} \), we have:

\[
a^2 + b^2 + c^2 \leq 4 \sum_{k=1}^{m} |\langle f, f_k \rangle|^2 = a^2 + b^2 + c^2 + (a + b + c)^2 \leq 4(a^2 + b^2 + c^2).
\]

So \( \{f_k\}_{k=1}^3 \) is a frame for \( \mathbb{R}^3 \) but not a basis of \( \mathbb{R}^3 \).

Now, we consider a vector space \( V \) equipped with a frame \( \{f_k\}_{k=1}^m \) and define a linear mapping

\[
T : \mathbb{C}^m \longrightarrow V, \\
\{c_k\}_{k=1}^m \longmapsto \sum_{k=1}^{m} c_k f_k.
\]

\( T \) is called pre-frame operator (or synthesis operator). The adjoint operator is given by

\[
T^* : V \longrightarrow \mathbb{C}^m, \\
f \longmapsto \{\langle f, f_k \rangle\}_{k=1}^{m},
\]
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Indeed, we have

\[ \langle T\{c_k\}_{k=1}^m, f \rangle = \left( \sum_{k=1}^m c_k f_k, f \right) = \sum_{k=1}^m c_k \langle f_k, f \rangle \]
\[ = \sum_{k=1}^m c_k \overline{f(f_k, f)} = \langle c_k, (f, f_k) \rangle \]
\[ = \langle c_k, T^* f \rangle. \]

\( T^* \) is called the **analysis operator**.

By composing \( T \) with its adjoint, \( T^* \), we obtain the **frame operator** \( S \):

\[ S : V \xrightarrow{T \circ T^*} V, \]
\[ f \mapsto \sum_{k=1}^m \langle f, f_k \rangle f_k. \]

So, we have \( \langle Sf, f \rangle = \sum_{k=1}^m |\langle f, f_k \rangle|^2 \) and

\[ A \|f\|^2 \leq \langle Sf, f \rangle \leq B \|f\|^2, \quad \forall f \in V. \tag{1.7} \]

Thus, the lower frame condition can be considered as "lower bound" of \( S \) and the upper frame bound can be considered as "upper bound" of \( S \).

**Definition 1.7** A frame \( \{f_k\}_{k=1}^m \) is tight if we can choose \( A = B \) in the Definition 1.1, i.e.,

\[ \sum_{k=1}^m |\langle f, f_k \rangle|^2 = A \|f\|^2, \quad \forall f \in V. \tag{1.8} \]

**Proposition 1.8** [1] Assume that \( \{f_k\}_{k=1}^m \) is a tight frame for \( V \) with frame bound \( A \). Then \( S = A \text{Id} \) (\( \text{Id} \) is the identity operator on \( V \)), and

\[ f = \frac{1}{A} \sum_{k=1}^m \langle f, f_k \rangle f_k \quad \forall f \in V. \tag{1.9} \]

In order to prove our proposition, we first prove the following lemma.

**Lemma 1.9** [1] Let \( \mathcal{H} \) be a Hilbert space. Consider \( U : \mathcal{H} \rightarrow \mathcal{H} \) be a bounded operator, and assume that \( \langle Ux, x \rangle = 0 \) for all \( x \in \mathcal{H} \). Then the following holds:
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(i) If $\mathcal{H}$ is a complex Hilbert space, then $U = 0$.

(ii) If $\mathcal{H}$ is a real Hilbert space and $U$ is self-adjoint, then $U = 0$.

Proof:

(i) If $\mathcal{H}$ is a complex Hilbert space then we have:

$$4\langle Ux, y \rangle = \langle U(x + y), x + y \rangle - \langle U(x - y), x - y \rangle + i\langle U(x + iy), x + iy \rangle -$$

$$- i\langle U(x - iy), x - iy \rangle, \forall x, y \in \mathcal{H}. \quad (1.10)$$

In detail, we have:

$$\langle U(x + y), x + y \rangle = \langle Ux, x \rangle + \langle Ux, y \rangle + \langle Uy, x \rangle + \langle Uy, y \rangle,$$

$$\langle U(x - y), x - y \rangle = \langle Ux, x \rangle - \langle Ux, y \rangle - \langle Uy, x \rangle + \langle Uy, y \rangle,$$

$$i\langle U(x + iy), x + iy \rangle = i\langle Ux, x \rangle - x^2 \langle Uy, y \rangle + i^2 \langle Uy, x \rangle - i^3 \langle Uy, y \rangle,$$

$$i\langle U(x - iy), x - iy \rangle = i\langle Ux, x \rangle + x^2 \langle Uy, y \rangle - i^2 \langle Uy, x \rangle - i^3 \langle Uy, y \rangle.$$

Then we obtain (1.10) by a direct calculation.

Since $\langle Ux, x \rangle = 0, \forall x \in \mathcal{H}$ then $\langle Ux, y \rangle = 0, \forall x, y \in \mathcal{H}$ by using (1.10).

That means $U = 0$.

(ii) If $\mathcal{H}$ is a real Hilbert space and $U$ is self-adjoint then let $\{e_k\}_{k=1}^\infty$ be an orthonormal basis for $\mathcal{H}$. For arbitrary $j, k \in \mathbb{N}$, we have:

$$0 = \langle U(e_k + e_j), e_k + e_j \rangle = \langle Ue_k, e_j \rangle + \langle Ue_j, e_k \rangle$$

$$= \langle Ue_k, e_j \rangle + \langle e_j, Ue_k \rangle = 2\langle Ue_k, e_j \rangle.$$

It implies $U = 0$. \hfill \blacksquare$

Now, we’re going to prove Proposition 1.8

Proof Let $U = S - AiD$, we have:

$$\langle Ux, y \rangle = \langle (S - AiD)x, y \rangle = \langle Sx, y \rangle - A\langle Idx, y \rangle$$

$$= \langle x, Sy \rangle - A\langle x, Idy \rangle$$

$$= \langle x, (S - AiD)y \rangle = \langle x, Uy \rangle.$$
Therefore, \( U \) is self adjoint. Applying Lemma 1.9, we get
\[
S - AId = 0, \text{ i.e., } S = AId,
\]
then
\[
Sf = Af, \quad \forall f \in V.
\]
Thus
\[
f = \frac{1}{A}Sf = \frac{1}{A} \sum_{k=1}^{m} \langle f, f_k \rangle f_k, \quad \forall f \in V.
\]

**Lemma 1.10** Let \( V, W \) be finite-dimensional vector spaces, equipped with inner products \( \langle ., . \rangle_V \), respectively \( \langle ., . \rangle_W \). Assume that \( \dim V = n, \dim W = m \).

Given a linear operator \( T : V \rightarrow W \), the adjoint operator \( T^* : W \rightarrow V \). Then the vector space \( \ker T \) and \( \operatorname{im} T^* \) are subspace of \( V \), and \( \ker T = \operatorname{im} T^* \perp \).

In particular, the linear map \( T \) induces orthogonal decomposition of \( V \) and (via \( T^* \) of \( W \)) given by
\[
V = \ker T \oplus \perp \operatorname{im} T^*
\]
\[
W = \ker T^* \oplus \perp \operatorname{im} T^*
\]

**Proof** Let \( v \in \ker T \), then \( Tv = 0 \).

For any \( u \in W \), we have: \( 0 = \langle Tv, u \rangle_W = \langle v, T^* u \rangle_V \). So \( v \in \operatorname{im} T^* \perp \).

Thus \( \ker T \subseteq \operatorname{im} T^* \perp \).

For \( v \in \operatorname{im} T^* \perp \), we have: \( \langle v, T^* u \rangle_V = 0, \forall u \in W \).

It is equivalent to
\[
\langle Tv, u \rangle_W = 0, \forall u \in W.
\]
Choose \( u = Tv \in W \), then \( \|Tv\|^2 = 0 \).

Therefore, \( Tv = 0 \) then \( v \in \ker T \). Thus, \( \ker T = \operatorname{im} T^* \perp \).

Similarly, we have \( \ker T^* = \operatorname{im} T \perp \). Applying Lemma 1.3 we have
\[
V = \ker T \oplus \perp \operatorname{im} T^*
\]
\[
W = \ker T^* \oplus \perp \operatorname{im} T.
\]
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An interpretation of (1.9) is that if \( \{f_k\}_{k=1}^m \) is a tight frame and we want to express \( f \in V \) as a linear combination \( f = \sum_{k=1}^m c_k f_k \), we can simply define \( g_k = \frac{1}{A} f_k \) and take \( c_k = \langle f, g_k \rangle \). Formula (1.9) is similar to the representation

\[
f = \sum_{k=1}^m \langle f, e_k \rangle e_k
\]

via an orthonormal basis \( \{e_k\}_{k=1}^m \); the only difference is the factor \( \frac{1}{A} \) in (1.9).

For general frames we now prove that we still have a representation of each \( f \in V \) of the form

\[
f = \sum_{k=1}^m \langle f, g_k \rangle f_k
\]

for an appropriate choice of \( \{g_k\}_{k=1}^m \). The obtained theorem is one of the most important results about frames, and (1.12) is called the frame decomposition:

**Theorem 1.11** [1] Let \( \{f_k\}_{k=1}^m \) be a frame for \( V \) with frame operator \( S \). Then

(i) \( S \) is invertible and self-adjoint.

(ii) Every \( f \in V \) can be represented as

\[
f = \sum_{k=1}^m \langle f, S^{-1} f_k \rangle f_k = \sum_{k=1}^m \langle f, f_k \rangle S^{-1} f_k.
\]

(iii) If \( f \in V \) also has the representation \( f = \sum_{k=1}^m c_k f_k \) for some scalar coefficients \( \{c_k\}_{k=1}^m \), then

\[
\sum_{k=1}^m |c_k|^2 = \sum_{k=1}^m |\langle f, S^{-1} f_k \rangle|^2 + \sum_{k=1}^m |c - \langle f, S^{-1} f_k \rangle|^2.
\]

**Proof**  
(i) We have \( S = T \circ T^* \) is self-adjoint. Now, we prove that \( S \) is bijective. Indeed, let \( f \in V \) and assume that \( Sf = 0 \) then

\[
0 = \langle Sf, f \rangle = \sum_{k=1}^m |\langle f, f_k \rangle|^2 \geq A \|f\|^2.
\]

Therefore, \( f = 0 \) then \( S \) is injective.

Moreover, since \( V = \text{span} \{f_k\}_{k=1}^m \) then \( T \) is surjective i.e., for \( f \in V \) there exists \( \{g_k\}_{k=1}^m \in \mathbb{C}^m \) such that \( T \{g_k\}_{k=1}^m = f \).

Rewrite

\[
\{g_k\}_{k=1}^m = \{f_k\}_{k=1}^m + \{h_k\}_{k=1}^m,
\]
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where \( \{ \overline{g}_k \}_{k=1}^m \in \text{Ker} T^⊥ \) and \( \{ h_k \}_{k=1}^m \in \text{Ker} T \).
Then

\[
f = T \{ g_k^m \}_{k=1}^m = T \{ \overline{g}_k \}_{k=1}^m + T \{ h_k \}_{k=1}^m = T \{ \overline{g}_k \}_{k=1}^m.
\]

By lemma 1.10, we have \( (\text{Ker} T)^⊥ = \text{Im} T^∗ \).
Therefore, for \( f \in V \) there exists \( \{ \overline{g}_k \}_{k=1}^m \in \text{Im} T^∗ \) such that \( T \{ \overline{g}_k \}_{k=1}^m = f \).
So, \( V \subset T(\text{Im} T^*) = \text{Im}(TT^*) = \text{Im} S \).
Thus, \( S \) is surjective. Hence, \( S \) is invertible.

(ii) For any \( u, v \in V \), \( S \) is self-adjoint then

\[
\langle S^{-1}u, v \rangle = \langle S^{-1}u, S \circ S^{-1}v \rangle = \langle S \circ S^{-1}u, S^{-1}v \rangle = \langle u, S^{-1}v \rangle.
\]
So \( S^{-1} \) is also self-adjoint. Therefore,

\[
f = S \circ S^{-1}f = \sum_{k=1}^m \langle S^{-1}f, f_k \rangle f_k = \sum_{k=1}^m \langle f, S^{-1}f_k \rangle f_k.
\]

Moreover,

\[
f = S^{-1} \circ Sf = S^{-1} \left( \sum_{k=1}^m \langle f, f_k \rangle f_k \right) = \sum_{k=1}^m \langle f, f_k \rangle S^{-1}f_k.
\]

(iii) We can write

\[
\{ c_k \}_{k=1}^m = \{ c_k \}_{k=1}^m - \{ \langle f, S^{-1}f_k \rangle \}_{k=1}^m + \{ \langle f, S^{-1}f_k \rangle \}_{k=1}^m.
\]

Since

\[
\sum_{k=1}^m \left( c_k - \langle f, S^{-1}f_k \rangle \right) f_k = \sum_{k=1}^m c_k f_k - \sum_{k=1}^m \langle f, S^{-1}f_k \rangle f_k = 0.
\]

then

\[
\left\{ \begin{array}{l}
\{ c_k \}_{k=1}^m - \{ \langle f, S^{-1}f_k \rangle \}_{k=1}^m \in \text{Ker} T = \text{Im} T^⊥, \\
\{ \langle f, S^{-1}f_k \rangle \}_{k=1}^m = \{ \langle S^{-1}f, f_k \rangle \}_{k=1}^m \in \text{Im} T^∗.
\end{array} \right.
\]

Thus, by Pythagoras’s Theorem, we get

\[
\sum_{k=1}^m |c_k|^2 = \sum_{k=1}^m |\langle f, S^{-1}f_k \rangle|^2 + \sum_{k=1}^m |c - \langle f, S^{-1}f_k \rangle|^2.
\]
1. Some basic facts about frames

Remark 1.12

- Every frame in a finite-dimensional space contains a subfamily which is a basic. Because \( \text{span}\{f_k\}_{k=1}^m = V \).

- If \( \{f_k\}_{k=1}^m \) is a frame but not a basis. Then there exist \( \{d_k\}_{k=1}^m \) non-zero sequence such that
  \[
  f = \sum_{k=1}^m \langle f, S^{-1}f_k \rangle f_k + \sum_{k=1}^m d_k f_k = \sum_{k=1}^m (\langle f, S^{-1}f_k \rangle + d_k ) f_k.
  \]
  It implies that \( f \) has many representations.

- Theorem 1.11 shows that \( \{\langle f, S^{-1}f_k \rangle\}_{k=1}^m \) has minimal \( \ell^2 \)-norm among all sequences \( \{c_k\}_{k=1}^m \) for which \( f = \sum_{k=1}^m c_k f_k \).
  The numbers \( \langle f, S^{-1}f_k \rangle, k = 1, 2, 3, ..., m \), are called frame coefficients.

- \( \{S^{-1}f_k\}_{k=1}^m \) is also a frame for \( V \) because \( \text{span}\{S^{-1}f_k\}_{k=1}^m = V \). It is called the canonical dual of \( \{f_k\}_{k=1}^m \). Theorem 1.11 gives some special information in case \( \{f_k\}_{k=1}^m \) is a basis:

Proposition 1.13 [1] Assume that \( \{f_k\}_{k=1}^m \) is a basis for \( V \). Then there exist a unique family \( \{g_k\}_{k=1}^m \) in \( V \) such that
  \[
  f = \sum_{k=1}^m \langle f, g_k \rangle f_k, \forall f \in V.
  \] (1.14)
  \[\text{In term of the frame operator, } \{g_k\}_{k=1}^m = \{S^{-1}f_k\}_{k=1}^m. \text{ Furthermore, } \langle f, g_k \rangle = \delta_{jk}. \]

**Proof** The existence of \( \{g_k\}_{k=1}^m \) follow from Theorem 1.11. Now, we suppose that there exist \( \{g'_k\}_{k=1}^m \) satisfying
  \[
  f = \sum_{k=1}^m \langle f, g'_k \rangle f_k = \sum_{k=1}^m \langle f, g_k \rangle f_k.
  \]
  Then,
  \[
  \sum_{k=1}^m \langle f, g'_k - g_k \rangle f_k = 0.
  \]
  Since \( \{f_k\}_{k=1}^m \) is a basis for \( V \) then \( \langle f, g'_k - g_k \rangle = 0, \forall f \in V \).
  Therefore, \( g'_k = g_k, \forall k = 1, 2, ..., m \) or \( \{g_k\}_{k=1}^m \) is unique!
1. Some basic facts about frames

Moreover, since \( \{ f_k \}_{k=1}^m \) is a basis and \( f_j = \sum_{k=1}^{m} \langle f_j, g_k \rangle f_k \), then

\[
\begin{align*}
\langle f_j, g_k \rangle &= 1 & \text{if } j = k \\
\langle f_j, g_k \rangle &= 0 & \text{if } j \neq k
\end{align*}
\]

or \( \langle f_j, g_k \rangle = \delta_{jk} \).

We can give an intuitive explanation of why frames are important in signal transmission. Assume that we want to transmit the signal \( f \) belonging to a vector space \( V \) from a transmitter \( A \) to a receiver \( R \). If both \( A \) and \( R \) have knowledge of a frame \( \{ f_k \}_{k=1}^m \) for \( V \), this can be done if \( A \) transmits the frame coefficients \( \{ \langle f_j, S^{-1} f_k \rangle \}_{k=1}^m \); based on knowledge of these numbers, the receiver \( R \) can reconstruct the signal \( f \) using the frame decomposition. Now assume that \( R \) receives a noisy signal, meaning a perturbation \( \{ \langle f_j, S^{-1} f_k \rangle + c_k \}_{k=1}^m \) of the correct frame coefficients. Based on the received coefficients, \( R \) will claim that the transmitted signal was

\[
\sum_{k=1}^{m} (\langle f_j, S^{-1} f_k \rangle + c_k) f_k = \sum_{k=1}^{m} (\langle f, S^{-1} f_k \rangle) f_k + \sum_{k=1}^{m} c_k f_k
\]

\[
= f + \sum_{k=1}^{m} c_k f_k.
\]

This differs from the correct signal \( f \) by the noise \( \sum_{k=1}^{m} c_k f_k \). If \( \{ f_k \}_{k=1}^m \) is overcomplete, the pre-frame operator \( T \{ c_k \}_{k=1}^m = \sum_{k=1}^{m} c_k f_k \) has a non-trivial kernel, implying that parts of the noise contribution might add up to zero and cancel. This will never happen if \( \{ f_k \}_{k=1}^m \) is an orthonormal basis! In that case

\[
\| \sum_{k=1}^{m} c_k f_k \|^2 = \sum_{k=1}^{m} |c_k|^2,
\]

so each noise contribution will make the reconstruction worse.

We have already seen that for given \( f \in V \), the frame coefficients \( \{ \langle f, S^{-1} f_k \rangle \}_{k=1}^m \) have minimal \( \ell^2 \)-norm among all sequences \( \{ c_k \}_{k=1}^m \) for which \( f = \sum_{k=1}^{m} c_k f_k \). We can also choose to minimize the norm in other space than \( \ell^2 \), we now show the existence of coefficients minimizing the \( \ell^1 \)-norm.

**Theorem 1.14** [1] *Let \( \{ f_k \}_{k=1}^m \) be a frame for a finite-dimensional vector space \( V \). Given \( f \in V \), there exist coefficients \( \{ d_k \}_{k=1}^m \) in \( \mathbb{C}^m \) such that \( f = \sum_{k=1}^{m} d_k f_k \) and

\[
\sum_{k=1}^{m} |d_k| = \inf \{ \sum_{k=1}^{m} |c_k| : f = \sum_{k=1}^{m} c_k f_k \}.
\]

(1.15)*
1. Some basic facts about frames

**Proof** Fix $f \in V$, it's clear that we can choose a set of coefficients \( \{ c_k \}_{k=1}^m \) such that $f = \sum_{k=1}^m c_k f_k$.

Let

$$r := \sum_{k=1}^m |c_k|,$$

and

$$M := \{ \{ d_k \}_{k=1}^m \in \mathbb{C}^m, \sum_{k=1}^m |d_k| \leq r ; k = 1, 2, ..., m \}.$$ 

$M$ is closed and bounded in finite-dimensional space $\mathbb{C}^m$, so $M$ is compact. Then the following closed set

$$N = \{ \{ d_k \}_{k=1}^m \in M | f = \sum_{k=1}^m d_k f_k \}$$

is also compact.

Consider

$$\phi : \mathbb{C} \rightarrow \mathbb{R},$$

$$\{ d_k \}_{k=1}^m \mapsto \sum_{k=1}^m |d_k|.$$ 

We have

$$|\phi(\{ c_k \}_{k=1}^m) - \phi(\{ d_k \}_{k=1}^m) | = \sum_{k=1}^m |(|c_k| - |d_k|)|$$

$$\leq \sum_{k=1}^m |c_k - d_k| = \phi(\{ c_k \}_{k=1}^m - \{ d_k \}_{k=1}^m).$$ 

So, $\phi(\{ c_k \}_{k=1}^m)$ tends to $\phi(\{ d_k \}_{k=1}^m)$ as $\{ c_k \}_{k=1}^m \rightarrow \{ d_k \}_{k=1}^m$.

Thus, $\phi$ is continuous.

Applying Weierstrass’s theorem, we get $\phi$ has minimum on $N$. In other words, there exist coefficients $\{ d_k \}_{k=1}^m$ in $\mathbb{C}^m$ such that $f = \sum_{k=1}^m d_k f_k$ and

$$\sum_{k=1}^m |d_k| = \inf \{ \sum_{k=1}^m |c_k| : f = \sum_{k=1}^m c_k f_k \}.$$
1. Some basic facts about frames

There are some important differences between Theorem 1.11 and Theorem 1.14. In Theorem 1.11 we find the sequence minimizing the $\ell^2$-norm of the coefficients in the expansion of $f$ explicitly; it is unique, and it depends linearly on $f$. On the other hand, Theorem 1.14 only gives the existence of an $\ell^1$-minimizer, and it may not be unique.

**Example 1.15**

Let $\{e_k\}_{k=1}^2$ be an orthonormal basis for two-dimensional vector space $V$ with $\langle \cdot, \cdot \rangle$.

Let

$$f_1 = e_1, \quad f_2 = e_1 - e_2, \quad f_3 = e_1 + e_2.$$ 

Then $\{f_k\}_{k=1}^3$ is a frame for $V$. Indeed, for $f \in V$, $f$ can be represent as

$$f = c_1e_1 + c_2e_2 \quad \text{for some scalar} \quad c_1, c_2 \in \mathbb{R}.$$

We have

$$\sum_{k=1}^3 |\langle f, f_k \rangle|^2 = |c_1|^2 + |c_1 - c_2|^2 + |c_1 + c_2|^2$$

$$= 3c_1^2 + 2c_2^2.$$

So,

$$2\|f\|^2 \leq \sum_{k=1}^3 |\langle f, f_k \rangle|^2 \leq 3\|f\|^2 \quad \forall f \in V.$$

Assume $f = \sum_{k=1}^3 d_k f_k$ then

$$\begin{cases} d_1 + d_2 + d_3 = c_1 \\ -d_2 + d_3 = c_2. \end{cases}$$

This equation system has at least 2 solutions $\{d_k\}_{k=1}^3$ that minimizing $\sum_{k=1}^3 |d_k|$.

Indeed, take $f = e_1$ then

$$\begin{cases} d_1 + d_2 + d_3 = 1 \\ -d_2 + d_3 = 0. \end{cases}$$
1. Some basic facts about frames

So, we can choose

\[
\begin{align*}
  d_1 &= 1 \\
  d_2 &= d_3 = 0,
\end{align*}
\]

or

\[
  d_1 = d_2 = d_3 = \frac{1}{3},
\]

then both cases get

\[
\sum_{k=1}^{3} |d_k| = 1.
\]

Even if the minimizer is unique, it may not depend linearly on \( f \).

**Example 1.16**

Let \( \{e_1, e_2\} \) be the canonical orthonormal basis for \( \mathbb{C}^2 \) and consider the frame \( \{f_k\}_{k=1}^{3} = \{e_1, e_2, e_1 + e_2\} \).

Then

\[
\begin{align*}
  \{c_k^{(1)}\}_{k=1}^{3} &= \{1, 0, 0\}, \\
  \{c_k^{(2)}\}_{k=1}^{3} &= \{0, 1, 0\},
\end{align*}
\]

minimize the \( \ell^1 \)-norm in the representation of \( e_1 \) and \( e_2 \), respectively.

Clearly, \( e_1 + e_2 = \sum_{k=1}^{3} (c_k^{(1)} + c_k^{(2)}) f_k \) but \( \{c_k^{(1)} + c_k^{(2)}\}_{k=1}^{3} \) not minimizing the \( \ell^1 \)-norm among all sequences representing \( e_1 + e_2 \).

**Theorem 1.17** \([1]\) Let \( \{f_k\}_{k=1}^{m} \) be a frame for a subspace \( W \) of the vector space \( V \). Then the orthogonal projection of \( V \) onto \( W \) is given by

\[
P f = \sum_{k=1}^{m} \langle f, S^{-1} f_k \rangle f_k, \quad f \in V.
\]

**Proof** It is enough to proof that if we define \( P \) by (1.16) then

\[
P f = \begin{cases} 
  f & \text{for } f \in W \\
  0 & \text{for } f \in W^\perp.
\end{cases}
\]

Because of Theorem 1.11, \( P f = f \) for \( f \in W \). Moreover, \( S \) is a bijection on \( W \) then \( S^{-1} f_k \in W \). It implies \( P f = 0 \) for \( f \in W^\perp \).
2. Frames in $\mathbb{C}^n$

The natural examples of finite-dimensional vector space are

$$\mathbb{R}^n = \{(c_1, c_2, \ldots, c_n) | c_i \in \mathbb{R}, i = 1, 2, \ldots, n\},$$

and

$$\mathbb{C}^n = \{(c_1, c_2, \ldots, c_n) | c_i \in \mathbb{C}, i = 1, 2, \ldots, n\}.$$

The latter is equipped with the inner product

$$\langle \{c_k\}_{k=1}^n, \{d_k\}_{k=1}^n \rangle = \sum_{k=1}^n c_k \overline{d_k},$$

and the associated norm

$$\|\{c_k\}_{k=1}^n\| = \left(\sum_{k=1}^n |c_k|^2\right)^{\frac{1}{2}}.$$

From elementary linear algebra, we know many equivalent conditions for a set of vectors to constitute a basis for $\mathbb{C}^n$. Let us list the most important characterizations:

**Theorem 1.18** [1] Consider $n$ vectors in $\mathbb{C}^n$ and write them as columns in an $n \times n$ matrix

$$\Lambda = \begin{pmatrix}
\lambda_{11} & \lambda_{12} & \ldots & \lambda_{1n} \\
\lambda_{21} & \lambda_{22} & \ldots & \lambda_{2n} \\
& & \ddots & \\
\lambda_{n1} & \lambda_{n2} & \ldots & \lambda_{nn}
\end{pmatrix}$$

then the following are equivalent:

(i) The columns in $\Lambda$ (i.e., the given vectors) constitute a basis for $\mathbb{C}^n$.

(ii) The rows in $\Lambda$ constitute a basis for $\mathbb{C}^n$.

(iii) The determinant of $\Lambda$ is non-zero.

(iv) $\Lambda$ is invertible.

(v) $\Lambda$ defines an injective mapping on $\mathbb{C}^n$. 

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2. Frames in $\mathbb{C}^n$

**(vi)** $\Lambda$ defines a surjective mapping on $\mathbb{C}^n$.

**(vii)** The columns in $\Lambda$ are linearly independent.

**(viii)** $\Lambda$ has rank equal to $n$.

**Proof (vii) $\leftrightarrow$ (i) $\leftrightarrow$ (iii)**

If the columns in $\Lambda$ are linearly independent then it is maximal linearly independent. So, it constitute a basis for $\mathbb{C}^n$.

If the columns in $\Lambda$ are linearly dependent then the determinant of $\Lambda$ should be zero.

Assume the columns in $\Lambda$ are linearly independent then it constitute a basis for $\mathbb{C}^n$.

Denote:

- $e = (e_1, e_2, ..., e_n)$ be the canonical basis for $\mathbb{C}^n$
- $\alpha_1, \alpha_2, ..., \alpha_n$ is the column vectors in $\Lambda$
- and, $\Lambda^n(\mathbb{C}^n)^*$ be the set of all alternative $n$-linear in $\mathbb{C}^n$.

So,

$$\det \Lambda = \det_e (\alpha_1, \alpha_2, ..., \alpha_n) \quad \text{where} \quad \alpha_j = \sum_{i=1}^{n} \lambda_{ij} e_i.$$  

Therefore, $\det = \det_e$ constitute a basis for $\Lambda^n(\mathbb{C}^n)^*$. We get

$$\det_e = c \det , \quad (c \in \mathbb{C}).$$

Thus

$$c \det(\alpha_1, \alpha_2, ..., \alpha_n) = \det_e (\alpha_1, \alpha_2, ..., \alpha_n) = \det I_n = 1.$$  

Hence,

$$\det \Lambda = \det(\alpha_1, \alpha_2, ..., \alpha_n) \neq 0.$$  

**(iii) $\leftrightarrow$ (ii)**

We have

$$\det \Lambda = \det \Lambda^T , \quad \forall \Lambda \in \mathbb{M}(n \times n, \mathbb{C}).$$

Then $\det \Lambda^T \neq 0$, it is equivalent to the columns of $\Lambda^T$ constitute a basis for $\mathbb{C}^n$ as the above proof.
Therefore, the rows of $\Lambda$ constitute a basis for $\mathbb{C}^n$.

(iii) $\leftrightarrow$ (iv)
Suppose $\Lambda$ define a homomorphism $f : \mathbb{C}^n \rightarrow \mathbb{C}^n$. Then we have to prove that $f$ is an isomorphism if $\det \Lambda \neq 0$.
Assume that $a = (a_1, a_2, ..., a_n)$ is a basis for $\mathbb{C}^n$. Then

$$(f(a_1) \ f(a_2) \ ... \ f(a_n)) = (a_1 \ a_2 \ ... \ n)\Lambda.$$

We have

$$\det \Lambda = \det(f) = \det(f(a_1) \ f(a_2) \ ... \ f(a_n)) = \det(f(a_1) \ f(a_2) \ ... \ f(a_n)) \det \text{Id}_n$$

$$= \det(f(a_1) \ f(a_2) \ ... \ f(a_n)) \det a(a_1, a_2, ..., n)$$

$$= \det a(f(a_1) \ f(a_2) \ ... \ f(a_n)) \det \Lambda.$$

Note that, $f$ is an isomorphism if and only if $(f(a_1), f(a_2), ..., f(a_n))$ is linearly independent.
This equivalent to $(a_1, a_2, ..., a_n)$ are linearly independent, or $\det \Lambda = \det(f) \neq 0$.

(iv) $\leftrightarrow$ (v) $\leftrightarrow$ (vi)
$\Lambda$ is invertible means that $\Lambda$ defines a bijective mapping on $\mathbb{C}^n$. Since $\mathbb{C}^n$ is finite then $\Lambda$ also defines an injective and a surjective mapping on $\mathbb{C}^n$.

(viii) $\leftrightarrow$ (vi)
The rank of $\Lambda$ should be the dimension of its range $\text{Im}\Lambda$. So, it must be $n$.

Recall that the rank of matrix $E$ is defines as the dimension of its range $\text{Im}E$.
Now, we turn to a discussion of frames for $\mathbb{C}^n$. Note that we consequently identify operator $A : \mathbb{C}^n \rightarrow \mathbb{C}^m$, with their matrix representations with respect to the canonical bases in $\mathbb{C}^n$ and $\mathbb{C}^m$.
In case $\{f_k\}_{k=1}^m$ is a frame for $\mathbb{C}^n$, the pre-frame operator $T$ maps $\mathbb{C}^m$ onto $\mathbb{C}^n$, and its matrix with respect to the canonical bases in $\mathbb{C}^n$ and $\mathbb{C}^m$ is

$$T = \begin{pmatrix} f_1 & f_2 & \cdots & f_n \end{pmatrix}$$

i.e., the $m \times n$ matrix having the vectors $f_k$ as columns.
Since $m$ vectors can at most span an $m$-dimensional space then we necessarily have $m \geq n$ when $\{f_k\}_{k=1}^m$ is a frame for $\mathbb{C}^n$, i.e., the matrix $T$ has at least as many columns as rows.
Theorem 1.19 [1] Let \( \{f_k\}_{k=1}^m \) be a frame for \( \mathbb{C}^n \). Then the vectors \( f_k \) can be considered as first \( n \) coordinates of some vectors \( g_k \) in \( \mathbb{C}^m \) constituting a basis for \( \mathbb{C}^m \). If \( \{f_k\}_{k=1}^m \) is a tight frame then the vectors \( f_k \) are the first \( n \) coordinates of some vector \( g_k \) in \( \mathbb{C}^m \) constituting an orthogonal basis for \( \mathbb{C}^m \).

PROOF Since \( \{f_k\}_{k=1}^m \) be an arbitrary frame for \( \mathbb{C}^n \) then

\[ \mathbb{C}^n = \text{span}\{f_k\}_{k=1}^m. \]

Therefore, \( m \geq n \).

Now, consider the pre-frame operator

\[ T : \mathbb{C}^m \rightarrow \mathbb{C}^n, \]

\[ \{c_k\}_{k=1}^m \mapsto \sum_{k=1}^m c_k f_k \]

corresponding to the matrix

\[ T = \begin{pmatrix} f_1 & f_2 & \cdots & f_n \end{pmatrix}. \]

Then the adjoint operator

\[ F : \mathbb{C}^n \rightarrow \mathbb{C}^m, \]

\[ x \mapsto \{\langle x, f_k \rangle\}_{k=1}^m \]

corresponding to

\[ F = \begin{pmatrix} -f_1 & - & - \\ - & -f_2 & - \\ - & - & \cdots \\ - & - & - & -f_n \end{pmatrix}. \]

We claim that \( F \) is injective. Indeed, if \( Fx = 0 \) then

\[ 0 = \|Fx\|^2 = \sum_{k=1}^m |\langle x, f_k \rangle|^2. \]

Therefore, \( |\langle x, f_k \rangle| = 0 \) for all \( k = 1, m \).

In addition, \( \text{span}\{f_k\}_{k=1}^m = \mathbb{C}^n \). Hence we get \( x = 0 \).
2. Frames in $\mathbb{C}^n$

For $x = (x_1, x_2, ..., x_n)^T \in \mathbb{C}^n$

$$Fx = \begin{pmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{m1} & f_{m2} & \cdots & f_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^n f_{1i}x_i \\ \sum_{i=1}^n f_{2i}x_i \\ \vdots \\ \sum_{i=1}^n f_{mi}x_i \end{pmatrix} = \begin{pmatrix} \langle x, f_1 \rangle \\ \langle x, f_2 \rangle \\ \vdots \\ \langle x, f_m \rangle \end{pmatrix}.$$  

Now, we extend $F$ to a bijection $\tilde{F}$ of $\mathbb{C}^m$ onto $\mathbb{C}^m$.

Let $\{e_k\}_{k=1}^m$ be the canonical basis for $\mathbb{C}^m$ and $\{\alpha_k\}_{k=n+1}^m$ be the orthogonal basis for $\text{Im}F^\perp$ in $\mathbb{C}^m$. Then the operator $\tilde{F}$ corresponding to the matrix

$$\tilde{F} = \begin{pmatrix} f_{11} & f_{12} & \cdots & f_{1n} & \alpha_1^{n+1} & \alpha_1^{n+2} & \cdots & \alpha_1^m \\ f_{21} & f_{22} & \cdots & f_{2n} & \alpha_2^{n+1} & \alpha_2^{n+2} & \cdots & \alpha_2^m \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ f_{m1} & f_{m2} & \cdots & f_{mn} & \alpha_m^{n+1} & \alpha_m^{n+2} & \cdots & \alpha_m^m \end{pmatrix}$$

is surjective.

Indeed, since $\text{span}\{f_k\}_{k=1}^n = \text{Im}F$, and $\text{span}\{\alpha_k\}_{k=n+1}^m = \text{Im}F^\perp$. Then the columns of $\tilde{F}$ spans $\mathbb{C}^m$.

Hence, the columns of $\tilde{F}$ constitute a basis for $\mathbb{C}^m$.

Applying Theorem 1.18, the rows of $\tilde{F}$ constitute a basis for $\mathbb{C}^m$. That means $f_k$ can be consider as the first $n$ coordinates of some vectors $g_k$ in $\mathbb{C}^m$ constituting a basis for $\mathbb{C}^m$.

If $\{f_k\}_{k=1}^m$ is a tight frame then rewrite

$$f = \frac{1}{A} \sum_{k=1}^m \langle f, f_k \rangle f_k$$

or $S = A\text{Id}$. So,

$$\langle TT^*e_i, e_j \rangle = A\delta_{ij}, \quad \forall i, j = 1, 2, ..., m.$$  

Thus, the $n$ rows in the matrix

$$T = \begin{pmatrix} {\mid} & \{ {\mid} \\ f_1 & f_2 & \cdots & f_n \end{pmatrix}$$

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are orthogonal.

By adding \( m - n \) rows, we can extend the matrix for \( T \) to an \( m \times n \) matrix in which the rows are orthogonal.

Therefore, the columns are orthogonal.

From Theorem 1.19 we can construct a frame or a tight frame of \( \mathbb{C}^n \) by taking a basis or an orthogonal basis of \( \mathbb{C}^m, m > n \), respectively. Then by taking first \( n \) coordinates, we have a frame, or a tight frame.

**Example 1.20**

Consider

\[
\begin{pmatrix}
1 \\
1 \\
1 \\
-1
\end{pmatrix}, \quad \begin{pmatrix}
1 \\
1 \\
-1 \\
1
\end{pmatrix}, \quad \begin{pmatrix}
1 \\
-1 \\
1 \\
1
\end{pmatrix}, \quad \begin{pmatrix}
-1 \\
1 \\
1 \\
1
\end{pmatrix}
\]

in \( \mathbb{C}^4 \).

By Theorem 1.19

\[
\begin{pmatrix}
1 \\
1 \\
1 \\
-1
\end{pmatrix}, \quad \begin{pmatrix}
1 \\
1 \\
-1 \\
1
\end{pmatrix}, \quad \begin{pmatrix}
1 \\
-1 \\
1 \\
1
\end{pmatrix}, \quad \begin{pmatrix}
-1 \\
1 \\
1 \\
1
\end{pmatrix}
\]

is a tight frame for \( \mathbb{C}^3 \) with frame bound \( A = 4 \).

For a given \( m \times n \) matrix \( \Lambda \) the following proposition gives a condition for the rows constituting a frame for \( \mathbb{C}^n \).

**Proposition 1.21** \([1]\) For an \( m \times n \) matrix

\[
\Lambda = \begin{pmatrix}
\lambda_{11} & \lambda_{12} & \ldots & \lambda_{1n} \\
\lambda_{21} & \lambda_{22} & \ldots & \lambda_{2n} \\
\vdots & \ddots & \ddots & \vdots \\
\lambda_{m1} & \lambda_{m2} & \ldots & \lambda_{mn}
\end{pmatrix}
\]

The following are equivalent:

(i) There exists a constant \( A > 0 \) such that

\[
A \sum_{k=1}^{n} |c_k|^2 \leq \| \Lambda \{c_k\}_k \|^2 \text{ for all } \{c_k\}_k \subseteq \mathbb{C}^n.
\]
(ii) The columns in \( \Lambda \) constitute a basis for their span in \( \mathbb{C}^m \).

(iii) The rows in \( \Lambda \) constitute a frame for \( \mathbb{C}^n \).

**Proof (i) \( \leftrightarrow \) (ii)**

Denote the columns in \( \Lambda \) by \( g_1, g_2, \ldots, g_n \); \( g_i \in \mathbb{C}^m \). Then (i) is equivalent to

\[
A \sum_{k=1}^{n} |c_k|^2 \leq \| \sum_{k=1}^{n} c_k g_k \|^2.
\]

So, by Theorem [1.4], \( \{g_k\}_{k=1}^{n} \) being a frame for their span. Moreover, \( \{g_k\}_{k=1}^{n} \) are linearly independent. Indeed, suppose that there exist \( \{\alpha_k\}_{k=1}^{n} \) and at least \( \alpha_i \neq 0 \) such that

\[
\sum_{k=1}^{n} \alpha_k g_k = 0.
\]

Then

\[
A \sum_{k=1}^{n} |c_k|^2 = 0
\]

implies \( \alpha_k = 0 \), \( \forall k = 1, n \).

Therefore, \( \{g_k\}_{k=1}^{n} \) being a basis for their span.

(i) \( \leftrightarrow \) (iii)

Now, denote the rows in \( \Lambda \) by \( f_1, f_2, \ldots, f_m \); \( f_i \in \mathbb{C}^n \). Then (i) is equivalent to

\[
A \sum_{k=1}^{n} |c_k|^2 \leq \sum_{k=1}^{n} |\langle f_k, \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \rangle|^2.
\]

It is equivalent to \( \{f_k\}_{k=1}^{m} \) should be a frame for \( \mathbb{C}^n \) by Theorem [1.4].

As an illustration of Proposition [1.21], consider

\[
\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]
It is clear that the rows \( \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \) constitute a frame for \( \mathbb{C}^2 \).

The columns
\[
\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}
\]
constitute a basis for their span in \( \mathbb{C}^3 \), but their span is only two-dimensional subspace of \( \mathbb{C}^3 \).

**Corollary 1.22** [1] Let \( \Lambda \) be an \( m \times n \) matrix. Denote the columns by \( g_1, g_2, ..., g_n \) and the rows by \( f_1, f_2, ..., f_m \). Given \( A, B > 0 \), the vectors \( \{ f_k \}_{k=1}^m \) constitute a frame for \( \mathbb{C}^n \) with bounds \( A, B \) if and only if
\[
A \sum_{k=1}^n |c_k|^2 \leq \| \sum_{k=1}^n c_k g_k \|^2 \leq B \sum_{k=1}^n |c_k|^2, \quad \{ c_k \}_{k=1}^n \in \mathbb{C}^n. \quad (1.17)
\]

**Proof.** We have (1.17) is equivalent to
\[
A \sum_{k=1}^n |c_k|^2 \leq \left( \sum_{k=1}^n |\langle f_k, \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \rangle | \right)^2 \leq B \sum_{k=1}^n |c_k|^2, \quad \{ c_k \}_{k=1}^n \in \mathbb{C}^n.
\]

Therefore, the vectors \( \{ f_k \}_{k=1}^m \) constitute a frame for \( \mathbb{C}^n \) with bounds \( A, B \). \( \square \)

**Example 1.23**

Consider
\[
\begin{pmatrix} \sqrt{3} \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]
in \( \mathbb{C}^3 \).

Corresponding to these vectors we consider matrix
\[
\Lambda = \begin{pmatrix}
0 & \frac{\sqrt{2}}{\sqrt{3}} & \frac{\sqrt{2}}{\sqrt{3}} \\
0 & -\frac{1}{\sqrt{3}} & \frac{\sqrt{2}}{\sqrt{3}} \\
\frac{\sqrt{2}}{\sqrt{3}} & 0 & 0 \\
\frac{1}{\sqrt{3}} & 0 & 0 \\
\frac{\sqrt{2}}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\
\frac{\sqrt{2}}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\
\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\
\frac{1}{\sqrt{3}} & -1 & -1 \\
\frac{1}{\sqrt{3}} & -1 & -1 \\
\frac{1}{\sqrt{3}} & -1 & -1
\end{pmatrix}
\]
2. Frames in $\mathbb{C}^n$

We can check that the column $\{g_k\}_{k=1}^3$ are normalized orthogonal in $\mathbb{C}^5$ with length $\sqrt{\frac{5}{3}}$.

Therefore,

$$\| \sum_{k=1}^3 c_k g_k \|^2 = \frac{5}{3} \sum_{k=1}^3 |c_k|^2.$$ 

for all $c_1, c_2, c_3$ in $\mathbb{C}$. By Corollary 1.22 we conclude that the vectors define by (1.18) constitute a tight frame for $\mathbb{C}^3$ with frame bound $\frac{5}{3}$.

For later use we state a special case of Corollary 1.22:

**Corollary 1.24** ([1]) Let $\Lambda$ be an $m \times n$ matrix. Then the following are equivalent:

(i) $\Lambda^* \Lambda = \text{Id}_{n \times n}$, in which $\text{Id}_{n \times n}$ is the $n \times n$ identity matrix.

(ii) The columns $g_1, g_2, ..., g_n$ in $\Lambda$ constitute an orthonormal system in $\mathbb{C}^m$.

(iii) The rows $f_1, f_2, ..., f_m$ in $\Lambda$ constitute a tight frame for $\mathbb{C}^n$ with frame bound equal to 1.

**Proof** Let $g_1, g_2, ..., g_n$ be the columns of $\Lambda$. Then

$$\Lambda^* \Lambda = \text{Id}_{n \times n}$$

equivalent to

$$\begin{pmatrix}
- \overline{f_1} & - & - \\
- \overline{f_2} & - & - \\
& \ddots & \\
- \overline{f_n} & - & -
\end{pmatrix}
\begin{pmatrix}
| & | & \cdots & | \\
| & | & \cdots & | \\
& \ddots & \\
| & | & \cdots & |
\end{pmatrix}
= \begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 1
\end{pmatrix}.$$ 

Thus,

$$\begin{cases}
|g_i|^2 = 1, & \forall i = 1, n, \\
\langle g_i, g_j \rangle = 0, & \forall i \neq j.
\end{cases}$$

Hence, $g_1, g_2, ..., g_n$ constitute an orthonormal system in $\mathbb{C}^m$.

Applying the Proposition 1.21 and Corollary 1.22 then

$$\| \sum_{k=1}^n c_k g_k \|^2 = \sum_{k=1}^n |c_k|^2.$$ 

Consequently, $f_1, f_2, ..., f_m$ is a tight frame for $\mathbb{C}^n$ with frame bound equal to 1. ■
Example 1.25

Consider

\[
\Lambda = \begin{pmatrix}
0 & \frac{1}{\sqrt{3}} & \frac{\sqrt{2}}{\sqrt{5}} \\
0 & -\frac{1}{\sqrt{5}} & \frac{\sqrt{2}}{\sqrt{5}} \\
0 & \frac{\sqrt{3}}{\sqrt{5}} & 0 \\
\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{10}} \\
\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{10}}
\end{pmatrix}
\]

then

\[
\Lambda^* = \begin{pmatrix}
0 & 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{5}} & -\frac{1}{\sqrt{5}} & \frac{\sqrt{3}}{\sqrt{5}} & 0 & 0 \\
\frac{\sqrt{3}}{\sqrt{5}} & \frac{\sqrt{3}}{\sqrt{5}} & 0 & \frac{1}{\sqrt{10}} & \frac{1}{\sqrt{10}}
\end{pmatrix}.
\]

We can check

\[
\Lambda^*\Lambda = \begin{pmatrix}
1 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 1
\end{pmatrix}.
\]

Thus, the columns in \(\Lambda\) constitute an orthonormal system in \(\mathbb{C}^5\).

The rows in \(\Lambda\) constitute a tight frame for \(\mathbb{C}^3\) with frame bound equal 1.

3 The discrete Fourier transform

When working with frames and bases in \(\mathbb{C}^n\), one has to be particularly careful with the meaning of the notation. For example: \(f_k, g_k\) be vectors in \(\mathbb{C}^n\); \(c_k\) be scalars. In order to avoid confusion, we will change the notation slightly.

Define vectors \(e_k\) in \(\mathbb{C}^n\) by

\[
e_k(j) = \frac{1}{\sqrt{n}} e^{2\pi i (j-1) \frac{k-1}{n}} , \quad j = \overline{1,n}.
\]

(1.19)
That is
\[ e_k = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 \\ e^{2\pi i k/n} \\ e^{4\pi i k/n} \\ \vdots \\ e^{2\pi i (n-1)k/n} \end{pmatrix}. \]

**Theorem 1.26 [1]** The vectors \( \{e_k\}_{k=1}^n \) defined in (1.19) constitute an orthonormal basis for \( \mathbb{C}^n \).

**PROOF** It is easy to see that \( \|e_k\| = 1 \) for all \( k = 1, \ldots, n \). Moreover,
\[
\langle e_k, e_\ell \rangle = \frac{1}{n} \sum_{j=1}^n e^{2\pi i (j-1)k/n} e^{-2\pi i (j-1)\ell/n} = \frac{1}{n} \sum_{j=0}^{n-1} e^{2\pi i j k/\ell}.
\]

Using the formula
\[
(1 - x)(1 + x + \ldots + x^{n-1}) = 1 - x^n
\]
with \( x = e^{2\pi i j k/\ell} \). Then \( \langle e_k, e_\ell \rangle = 0 \) for all \( k \neq \ell \).

The basis \( \{e_k\}_{k=1}^n \) is called the **discrete Fourier transform** basis. Using this basis, every sequence \( f \in \mathbb{C}^n \) has a representation
\[
f = \sum_{k=1}^n \langle f, e_k \rangle e_k = \frac{1}{\sqrt{n}} \sum_{k=1}^n \sum_{\ell=1}^n f(\ell) e^{-2\pi i (\ell-1)k/n} e_k. \tag{1.20}
\]

Applications often ask for tight frames because the cumbersome inversion of the frame operator is avoided in this case, see (1.9). It is interesting that overcomplete tight frames can be obtained in \( \mathbb{C}^n \) by projecting the discrete Fourier transform basis in any \( \mathbb{C}^m \), \( m > n \), onto \( \mathbb{C}^n \):

**Proposition 1.27 [1]** Let \( m > n \) and define the vectors \( \{f_k\}_{k=1}^m \) in \( \mathbb{C}^n \) by
\[
f_k = \frac{1}{\sqrt{m}} \begin{pmatrix} 1 \\ e^{2\pi i k/m} \\ e^{4\pi i k/m} \\ \vdots \\ e^{2\pi i (n-1)k/m} \end{pmatrix}; \quad k = 1, 2, \ldots, m. \tag{1.21}
\]
3. The discrete Fourier transform

Then \( \{f_k\}_{k=1}^m \) is a tight overcomplete frame for \( \mathbb{C}^n \) with frame bound equal to 1 and \( \|f_k\| = \frac{\sqrt{n}}{m} \) for all \( k \).

**Proof** Applying Theorem 1.26, one has \( \{\tilde{f}_k\}_{k=1}^m \) constitute an orthonormal basis for \( \mathbb{C}^m \) with

\[
\tilde{f}_k = \frac{1}{\sqrt{m}} \begin{pmatrix}
1 \\
e^{2\pi i k - 1} \\
e^{4\pi i k - 1} \\
e^{2\pi i (m-1)(k-1)}
\end{pmatrix}; \quad k = 1, 2, ..., m. \tag{1.22}
\]

Applying the Corollary 1.24, \( \{f_k\}_{k=1}^m \) constitute a tight frame bound for \( \mathbb{C}^n \) with frame bound equal to 1.

In addition, \( \|f_k\| = \sqrt{\frac{n}{m}} \) for all \( k = 1, m \).

It is important to notice that all the vectors \( f_k \) in Proposition 1.27 have the same norm. If needed, we can therefore normalize them while keeping a tight frame; we only have to adjust the frame bound accordingly.

**Corollary 1.28** [1] For any \( m > n \), there exists a tight frame in \( \mathbb{C}^n \) consisting of \( m \) normalized vectors.

**Example 1.29**

The discrete Fourier transform basis in \( \mathbb{C}^4 \) consists of the vectors

\[
\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \\ -1 \\ -i \end{pmatrix}, \quad \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 1 \\ -i \end{pmatrix}, \quad \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \\ 1 \\ i \end{pmatrix}.
\]

Via Proposition 1.27, the vectors

\[
\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \\ -1 \\ -i \end{pmatrix}, \quad \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 1 \\ -i \end{pmatrix}, \quad \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \\ 1 \\ i \end{pmatrix}.
\]

constitute a tight frame in \( \mathbb{C}^3 \).

One advantage of an overcomplete frame compared to a basis is that the frame property might be kept if an element is removed. However, even for frames it can
happen that the remaining set is no longer a frame, for example if the removed element is orthogonal to the rest of the frame elements. Unfortunately this can be the case no matter how large the number of frame elements is, i.e., no matter how redundant the frame is! If we have information on the lower frame bound and the norm of the frame elements we can provide a criterion for how many elements we can (at least) remove:

**Proposition 1.30** [1] Let \( \{ f_k \}_{k=1}^m \) be a normalized frame for \( \mathbb{C}^n \) with lower bound \( A > 1 \). Then for any index set \( I \subset \{1,2,\ldots,m\} \) with \( |I| < A \), the family \( \{ f_k \}_{k \not\in I} \) is a frame for \( \mathbb{C}^n \) with lower frame bound \( A - |I| \).

**Proof** Given \( f \in \mathbb{C}^n \),
\[
\sum_{k \in I} |\langle f, f_k \rangle|^2 \leq \sum_{k \in I} \|f_k\|^2 \|f\|^2 = |I| \|f\|^2.
\]
Thus,
\[
\sum_{k \not\in I} |\langle f, f_k \rangle|^2 \geq (A - |I|) \|f\|^2.
\]
Since \( A - |I| > 0 \) then we can choose \( (A - |I|) \) as a lower frame bound of \( \{ f_k \}_{k=1}^m \).

Considering an arbitrary frame \( \{ f_k \}_{k=1}^m \) for \( \mathbb{C}^n \), the maximal number of elements one can hope to remove while keeping the frame property is \( m - n \). If we want to be able to remove \( m - n \) arbitrary elements it is not enough to assume that \( \{ f_k \}_{k=1}^m \) is a normalized tight frame. Concerning the stability against removal of vectors, the frames obtained in Proposition 1.27 behave well: \( m - n \) arbitrary elements can be removed:

**Proposition 1.31** [1] Consider the frame \( \{ f_k \}_{k=1}^m \) for \( \mathbb{C}^n \) define as (1.21). Any subset containing at least \( n \) elements of this frame forms a frame for \( \mathbb{C}^n \).

**Proof** Consider an arbitrary subset \( \{ k_1, k_2, \ldots, k_n \} \subset \{1,2,\ldots,m\} \). Placing the vectors \( \{ f_{k_i} \}_{i=1}^n \) as rows in an \( n \times n \) matrix and letting \( z := e^{2\pi i/m} \), then
\[
\left(\begin{array}{c}
- f_{k_1} \\
- f_{k_2} \\
\vdots \\
- f_{k_n}
\end{array}\right) = \frac{1}{\sqrt{m}} \left(\begin{array}{cccc}
1 & z^{k_1-1} & \ldots & z^{(k_1-1)(n-1)} \\
1 & z^{k_2-1} & \ldots & z^{(k_2-1)(n-1)} \\
\vdots & \vdots & \ddots & \vdots \\
1 & z^{k_n-1} & \ldots & z^{(k_n-1)(n-1)}
\end{array}\right).
\]
This is Vandermonde matrix with determinant
\[
\det = \frac{1}{m^n} \prod_{i \neq j} (z^{k_i-1} - z^{k_j-1}) \neq 0.
\tag{1.23}
\]

Hence, \( \{f_k\}_{i=1}^n \) is a basis for \( \mathbb{C}^n \).

From Proposition 1.31, we have another way to construct a frame for \( \mathbb{C}^n \). In detail, taking any at least \( n \) elements of the frame \( \{f_k\}_{k=1}^m \) define as (1.21), we will obtain a frame for \( \mathbb{C}^n \).

## 4 Pseudo-inverses and the singular value decomposition

Given \( m \times n \) matrix \( E \), consider it as a linear mapping
\[
E : \mathbb{C}^n \rightarrow \mathbb{C}^m.
\]

\( E \) is not necessary injective, but by restricting \( E \) to orthogonal complement of the kernel \( \text{Ker}E \), we obtain an injective linear mapping
\[
\tilde{E} : (\text{Ker}E)^\perp \rightarrow \mathbb{C}^m.
\]

Indeed, if \( \tilde{E}x = 0 \) then \( x = 0 \) because \( \text{Ker}E \cap (\text{Ker}E)^\perp = \{0\} \). So, \( \tilde{E} \) is injective. We see that \( \text{Im}E = \text{Im}\tilde{E} \) then
\[
\tilde{E} : (\text{Ker}E)^\perp \rightarrow \text{Im}E
\]
has inverse and its inverse
\[
(\tilde{E})^{-1} : \text{Im}E \rightarrow (\text{Ker}E)^\perp.
\]

Extend \( \tilde{E}^{-1} \) to \( E^\dagger : \mathbb{C}^m \rightarrow \mathbb{C}^n \) by defining
\[
E^\dagger(y + z) = \tilde{E}^{-1}y \text{ if } y \in \text{Im}E, \ z \in (\text{Im}E)^\perp.
\tag{1.24}
\]

So,
\[
EE^\dagger x = x \text{ for all } x \in \text{Im}E.
\]
4. Pseudo-inverses and the singular value decomposition

The operator $E^\dagger$ is called pseudo-inverse of $E$.

From the construction and Lemma 1.10, we have:

\[ \text{Ker} E^\dagger = \text{Im} E^\perp = \text{Ker} E^* \quad (1.25) \]
\[ \text{Im} E^\dagger = \text{Ker} E^\perp = \text{Im} E^* . \quad (1.26) \]

Indeed, if $x \in (\text{Im} E)^\perp$ then $E^\dagger x = 0$ by (1.24). So, $x \in \text{Ker} E^\dagger$.

Thus $(\text{Im} E)^\perp \subseteq \text{Ker} E^\dagger$.

If $x \in \text{Ker} E^\dagger$ then $0 = E^\dagger x = \tilde{E}^{-1} y$, for which $x = y + z$, $y \in \text{Im} E$, $z \in (\text{Im} E)^\perp$.

Since $\tilde{E}^{-1}$ is bijective then $y = 0$. Therefore, $x = z \in (\text{Im} E)^\perp$.

Thus, $\text{Ker} E^\dagger \subseteq (\text{Im} E)^\perp$.

In conclusion, we obtain

\[ \text{Ker} E^\dagger = \text{Im} E^\perp = \text{Ker} E^*. \]

Similarly, we have

\[ \text{Im} E^\dagger = \text{Ker} E^\perp = \text{Im} E^*. \]

We note two characterizations of the pseudo-inverse:

**Proposition 1.32** [1] *Let $E$ be an $m \times n$ matrix. Then*

(i) $E^\dagger$ is the unique $n \times m$ matrix for which $EE^\dagger$ is the orthogonal projection onto $\text{Im} E$ and $E^\dagger E$ is the orthogonal projection onto $1_{\text{Im} E^\dagger}$.

(ii) $E^\dagger$ is the unique $n \times m$ matrix for which $EE^\dagger$ and $E^\dagger E$ are self-adjoint and

\[ EE^\dagger E = E , \ E^\dagger EE^\dagger = E^\dagger . \]

**Proof** Assume that $EE^\dagger$ is the orthogonal projection onto $\text{Im} E$ then $EE^\dagger E = E$ and $EE^\dagger x = 0$, $\forall x \in (\text{Im} E)^\perp$. Therefore, $(\text{Im} EE^\dagger)^\perp = \text{Ker} EE^\dagger$, i.e., $EE^\dagger$ is self-adjoint.

Similarly, we have $E^\dagger E$ is self-adjoint and $E^\dagger EE^\dagger = E^\dagger$.

Suppose $EE^\dagger$ is self-adjoint and $EE^\dagger E = E$. Then, we have

\[ (EE^\dagger)^2 = EE^\dagger EE^\dagger = EE^\dagger . \]

So, $EE^\dagger$ is a projection onto $\text{Im} EE^\dagger$. Since $EE^\dagger$ is self-adjoint then

\[ (\text{Im} EE^\dagger)^\perp = \text{Ker} EE^\dagger, \text{ i.e., } EE^\dagger(x) = 0 , \forall x \in (\text{Im} EE^\dagger)^\perp. \]
Therefore, $EE^\dagger$ is the orthogonal projection onto $\text{Im} EE^\dagger$.
Moreover, $EE^\dagger E = E$ then $\text{Im} E = \text{Im} EE^\dagger E \subseteq \text{Im} EE^\dagger \subseteq \text{Im} E$.
Hence, $\text{Im} EE^\dagger = \text{Im} E$.
That means $EE^\dagger$ is the orthogonal projection onto $\text{Im} E$.
Similar for $E^\dagger E$, $E^\dagger E$ is the orthogonal projection onto $\text{Im} E^\dagger$.

Now, we have to prove that $E^\dagger$ in Proposition 1.32 is exact $E^\dagger$ in the above construction.

**Proof**
Assume $E^\dagger$ as in the construction, if $y \in \text{Im} E$ then $EE^\dagger y = y$.
If $y \in (\text{Im} E)^\perp = \text{Ker} E^\dagger$ then $EE^\dagger y = 0$.
Therefore, $EE^\dagger$ is the orthogonal projection onto $\text{Im} E$.
In addition, if $y \in (\text{Im} E^\dagger)^\perp = \text{Ker} E$ then $E^\dagger Ey = 0$.
If $y \in \text{Im} E^\dagger$ then $y = E^\dagger x$ for some $x$. So,
\[
E^\dagger Ey = E^\dagger EE^\dagger x = E^\dagger x - E^\dagger (\text{Id} - EE^\dagger)x = E^\dagger x = y,
\]
here $\text{Id} - EE^\dagger$ is the orthogonal projection onto $(\text{Im} E)^\perp = \text{Ker} E^\dagger$.
Thus, $E^\dagger Ey = \text{Im} E^\dagger$.
Hence, $E^\dagger E$ is the orthogonal projection onto $\text{Im} E^\dagger$.
Conversely, suppose $E^\dagger$ is defined as in Proposition 1.32.
From (ii)
\[
E^* = (EE^\dagger E)^* = (E^\dagger E)^* E^* = E^\dagger EE^*
\]
because $E^\dagger E$ is self-adjoint.
Therefore,
\[
(\text{Ker} E)^\perp = \text{Im} E^* \subseteq \text{Im} E^\dagger.
\]
Now, take $y \in \text{Im} E$ then $\exists x \in (\text{Ker} E)^\perp$ such that $y = Ex$. Then,
\[
E^\dagger y = E^\dagger Ex = x = (\tilde{E})^{-1}Ex = (\tilde{E})^{-1}y.
\]
If $z \in (\text{Im} E)^\perp$ then $EE^\dagger z = 0$ by (i),
and $E^\dagger z = E^\dagger EE^\dagger z = 0$ by (ii).

The pseudo-inverse gives the solution to an important minimization problem:
Theorem 1.33 [1] Let \( E \) be an \( m \times n \) matrix. Given \( y \in \text{Im} E \). Then the equation \( Ex = y \) has a unique solution of minimal norm, namely \( x = E^\dagger y \).

**Proof** We have: \( EE^\dagger y = y \) for \( y \in \text{Im} E \). So, \( E^\dagger y \) is a solution to the equation \( Ex = y \).

Thus, the general solution \( x = E^\dagger y + z \), where \( z \in \text{Ker} E \) and \( y \in (\text{Ker} E)^\perp \).

Hence, \( \|x\|^2 = \|E^\dagger y\|^2 + \|z\|^2 \geq \|E^\dagger y\|^2 \). It occurs when \( z = 0 \).

For computational purposes it is important to notice that the pseudo-inverse can be found using the singular value decomposition of \( E \). We begin with a lemma.

Lemma 1.34 [1] Let \( E \) be an \( m \times n \) matrix with rank \( r \geq 1 \). Then there exist constants \( \sigma_1, \sigma_2, \ldots, \sigma_r > 0 \) and orthonormal bases \( \{u_k\}_{k=1}^r \) for \( \text{Im} E \) and \( \{v_k\}_{k=1}^r \) for \( \text{Im} E^\ast \) such that

\[
Ev_k = \sigma_k u_k \text{ for } k = 1, r. \quad (1.27)
\]

**Proof** We have:

\[
E^\ast E : \mathbb{C}^n \rightarrow \mathbb{C}^n
\]

is self-adjoint.

Then there exists an orthonormal basis \( \{v_k\}_{k=1}^n \) for \( \mathbb{C}^n \) consisting of eigenvectors for \( E^\ast E \).

Let \( \{\lambda_k\}_{k=1}^n \) denote the corresponding eigenvalues. Then

\[
\lambda_k = \lambda_k \|v_k\|^2
= \langle E^\ast Ev_k, v_k \rangle = \|Ev_k\|^2 \geq 0, \quad \forall k = 1, n.
\]

Thus,

\[
r = \text{dim} \text{Im} \, E = \text{dim} \text{Im} \, E^\ast. \quad (1.28)
\]

Since

\[
(\text{Im} \, E)^\perp = \text{Ker} E^\ast
\]
then
\[ ImE^* = ImE^*E = \text{span}\{E^*Ev_k\}_{k=1}^n = \text{span}\{\lambda_kv_k\}_{k=1}^n. \]

Therefore, the rank equal to the number of non-zero eigenvalues. Assume that we can order \( \{v_k\}_{k=1}^r \) such that \( \{v_k\}_{k=1}^r \) corresponds to the non-zero eigenvalues. Since
\[ ImE^* = \text{span}\{\lambda_kv_k\}_{k=1}^r \]
then \( \{v_k\}_{k=1}^r \) is an orthonormal basis for \( ImE^* \).
For \( k > r \),
\[ \|Ev_k\|^2 = \langle E^*Ev_k, v_k \rangle = 0. \]
Then \( Ev_k = 0 \), \( \forall k > r \).
Define
\[ u_k := \frac{1}{\sqrt{\lambda_k}}Ev_k, \quad k = 1, 2, ..., r. \] (1.29)
Therefore, we obtain \( \{u_k\}_{k=1}^r \) spans \( ImE \).
Hence,
\[ \sigma_k = \sqrt{\lambda_k}, \quad k = 1, 2, ..., r. \]

Lemma 1.34 leads to the singular value decomposition of \( E \):

**Theorem 1.35 [1]** Every \( m \times n \) matrix \( E \) with rank \( r \geq 1 \) has a decomposition:
\[ E = U \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} V^* \] (1.30)
where \( U \) is a unitary \( m \times m \) matrix,
\( V \) is a unitary \( n \times n \) matrix,
and \( \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} \) is an \( m \times n \) block matrix in which \( D \) is an \( r \times r \) diagonal matrix with positive entries \( \sigma_1, \sigma_2, ..., \sigma_r \) in the diagonal.
The numbers $\sigma_1, \ldots, \sigma_r$ are called singular values of $E$; the proof of Lemma 1.34 shows that they are the square roots of the positive eigenvalues for $E^*E$.

**Corollary 1.36** [1] With the notation in Theorem 1.35, the pseudo-inverse of $E$ is given by

$$E^+ = V \begin{pmatrix} D^{-1} & 0 \\ 0 & 0 \end{pmatrix} U^*$$

(1.32)
where \( \begin{pmatrix} D^{-1} & 0 \\ 0 & 0 \end{pmatrix} \) is an \( m \times n \) block matrix in which \( D^{-1} \) is an \( r \times r \) diagonal matrix with positive entries \( \sigma_1^{-1}, \sigma_2^{-1}, \ldots, \sigma_r^{-1} \) in the diagonal.

**Proof** We check that the matrix \( E^\dagger \) defined by (1.38) satisfies the requirements in Proposition 1.32. First, via (1.30),

\[
EE^\dagger = U \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} V^* \begin{pmatrix} D^{-1} & 0 \\ 0 & 0 \end{pmatrix} U^* = U \begin{pmatrix} \text{Id} & 0 \\ 0 & 0 \end{pmatrix} U^*,
\]

which shows that \( EE^\dagger \) is self-adjoint. Furthermore,

\[
EE^\dagger E = U \begin{pmatrix} \text{Id} & 0 \\ 0 & 0 \end{pmatrix} U^* U \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} V^* = E.
\]

Similarly, we can verify that \( E^\dagger E \) is self-adjoint and \( E^\dagger EE^\dagger = E^\dagger \).

Let us return to the setting where \( \{f_k\}_{k=1}^m \) is a frame for \( \mathbb{C}^n \) with pre-frame operator \( T : \mathbb{C}^m \to \mathbb{C}^n \). The calculation of the frame coefficients amounts to finding \( T^\dagger \):

**Theorem 1.37** [1] Let \( \{f_k\}_{k=1}^m \) be a frame for \( \mathbb{C}^n \), with pre-frame operator \( T \) and frame operator \( S \). Then

\[
T^\dagger f = \{\langle f, S^{-1}f_k \rangle\}_{k=1}^m, \quad \forall f \in \mathbb{C}^m.
\]  

**Proof** Let \( f \in \mathbb{C}^n \). Expressed in terms of the pre-frame operator \( T \), the equation \( f = \sum_{k=1}^m c_k f_k \) means that \( T\{c_k\}_{k=1}^m = f, f \in \mathbb{C}^m \).

By Theorem 1.33, the equation

\[
T\{c_k\}_{k=1}^m = f
\]
has a unique solution of minimal norm \( \{c_k\}_{k=1}^m = T^+ f \).

Applying Theorem 1.11, we have:

\[
c_k = \langle f, S^{-1}k \rangle, \quad \forall k = 1, m.
\]

Therefore,

\[
T^+ f = \{\langle f, S^{-1}f_k \rangle\}_{k=1}^m, \quad \forall f \in \mathbb{C}^m.
\]

One interpretation of Theorem 1.37 is that when \( \{f_k\}_{k=1}^m \) is a frame for \( \mathbb{C}^n \), the matrix \( T^+ \) is obtained by replacing the complex conjugate of the vectors in the canonical dual frame \( \{S^{-1}f\}_{k=1}^m \) as rows in an \( m \times n \) matrix:

\[
T^+ = \begin{pmatrix}
- S^{-1}f_1 & - \\
- S^{-1}f_2 & - \\
\vdots & \\
- S^{-1}f_m & -
\end{pmatrix}.
\]

The singular value decomposition gives a natural way to obtain coefficients \( \{c_k\}_{k=1}^m \) such that \( f = \sum_{k=1}^m c_k f_k \). Let \( \{f_k\}_{k=1}^m \) be an overcomplete frame for \( \mathbb{C}^n \). Since \( T \) is surjective, its rank equals \( n \), and the singular value decomposition of \( T \) is

\[
T = U(D \quad 0)V^*.
\]

Note that since \( T \) is an \( n \times m \) matrix, \( (D \quad 0) \) is now an \( n \times m \) matrix; \( U \) is an \( n \times n \) matrix, and \( V \) is an \( m \times n \) matrix. Given any \( (m-n) \times n \) matrix \( F \), we have

\[
TV \begin{pmatrix} D^{-1} \\ F \end{pmatrix} U^* f = U(D \quad 0)V^* V \begin{pmatrix} D^{-1} \\ F \end{pmatrix} U^* f
\]

\[
= U I U^* f
\]

\[
= f.
\]

This means that we can use the coefficients

\[
\{c_k\}_{k=1}^m = V \begin{pmatrix} D^{-1} \\ F \end{pmatrix} U^* f
\]

(1.34)
for the reconstruction of $f$. As noted already in Theorem 1.11 the choice $F = 0$ is optimal in the sense that the $\ell^2$-norm of the coefficients is minimized, but for other purposes other choices might be preferable. The matrix

$$V \begin{pmatrix} D^\dagger & F \end{pmatrix} U^*$$  \hspace{1cm} (1.35)

is frequently called a *generalized inverse* of $T$.

### 5 Finite-dimensional function spaces

Given $a, b \in \mathbb{R}$ with $a < b$, let $C[a, b]$ denote the set of continuous functions $f$: $[a, b] \rightarrow \mathbb{C}$. We equip $C[a, b]$ with the supremums-norm

$$\|f\|_\infty = \sup_{x \in [a, b]} |f(x)|.$$  \hspace{1cm} (1.36)

The Weierstrass’s Approximation Theorem says that every $f \in C[a, b]$ can be approximated arbitrarily well by a polynomial:

**Theorem 1.38 [1]** Let $f \in C[a, b]$. Given $\varepsilon > 0$, there exists a polynomial

$$P(x) = \sum_{k=0}^{n} c_k x^k$$  \hspace{1cm} (1.37)

such that

$$\|f(x) - P(x)\|_\infty \leq \varepsilon \quad \text{for all} \quad x \in [a, b].$$  \hspace{1cm} (1.38)

**Proof** Let $t = a + x(b - a)$ for $t \in [a, b]$, then $x \in [0, 1]$. We will show that the polynomial

$$B_k(x) = B_k f(x) = \sum_{p=0}^{k} C_{k}^{p} f\left(\frac{B}{k}\right) x^p (1 - x)^{k-p}$$  \hspace{1cm} (1.39)

converges to $f$ for all $x \in [0, 1]$. Indeed, remark

$$(x + y)^k = \sum_{p=0}^{k} C_{k}^{p} x^p y^{k-p}.$$  \hspace{1cm} (1.40)
Taking the derivative (1.38) and multiplying $x$ to both sides

$$kx(x + y)^{k-1} = \sum_{p=0}^{k} pC_k^px^py^{k-p}.$$  

(1.39)

Taking the derivative twice (1.38) and multiplying $x^2$ to both sides

$$k(k-1)x^2(x + y)^{k-2} = \sum_{p=0}^{k} p(p-1)C_k^px^py^{k-p}.$$  

(1.40)

Let $y := 1 - x, r_p(x) = C_k^px^py^{k-p}$, then

$$\sum_{p=0}^{k} r_p(x) = 1, \sum_{p=0}^{k} pr_p(x) = kx, \sum_{p=0}^{k} p(p-1)r_p(x) = k(k-1)x^2.$$

Therefore,

$$\sum_{p=0}^{k} (p - kx)^2r_p(x) = \sum_{p=0}^{k} p^2r_p(x) - 2kx \sum_{p=0}^{k} pr_p(x) + k^2x^2 \sum_{p=0}^{k} r_p(x)$$

$$= [kx + (k-1)kx^2] - 2kx + k^2x^2$$

$$= kx(1 - x).$$

Now, let

$$M = \max_{|x| \leq 1} |f(x)|.$$  

(1.41)

Given $\epsilon > 0$, since $f \in C[a,b]$ then

$$\exists \delta > 0 \text{ such that } |x - y| < \delta \text{ yields } |f(x) - f(y)| < \epsilon, \forall x, y \in [a,b].$$  

(1.42)

Consider

$$f(x) - B_k(x) = f(x) - \sum_{p=0}^{k} C_k^p f\left(\frac{p}{k}\right)x^p(1 - x)^{k-p}$$

$$= \sum_{p=0}^{k} f(x)r_p(x) - \sum_{p=0}^{k} C_k^p f\left(\frac{p}{k}\right)x^p(1 - x)^{k-p}$$

$$= \sum_{p=0}^{k} (f(x) - f\left(\frac{p}{k}\right))r_p(x).$$
Thus,
\[ f(x) - B_k(x) = \sum_{\{p\|\frac{p}{k} - x\| < \delta\}} (f(x) - f\left(\frac{p}{k}\right))r_p(x) + \sum_{\{p\|\frac{p}{k} - x\| \geq \delta\}} (f(x) - f\left(\frac{p}{k}\right))r_p(x). \]

(1.43)

Since \( r_p(x) \geq 0 \) and 
\[ |f(x) - f\left(\frac{p}{k}\right)| < \frac{\epsilon}{2} \text{ when } \left|\frac{p}{k} - x\right| < \delta \]
then
\[ |\sum_{\{p\|\frac{p}{k} - x\| \geq \delta\}} (f(x) - f\left(\frac{p}{k}\right))r_p(x)| \leq \frac{\epsilon}{2} \sum_{p=0}^{k} r_p(x) = \frac{\epsilon}{2}. \]

Moreover,
\[ |\sum_{\{p\|\frac{p}{k} - x\| \geq \delta\}} (f(x) - f\left(\frac{p}{k}\right))r_p(x)| \leq 2M \sum_{\{p\|\frac{p}{k} - x\| \geq \delta\}} r_p(x) \]
\[ \leq 2M \sum_{p=0}^{k} \frac{(p - k\delta)^2}{k\delta} r_p(x) \]
\[ = \frac{2M}{k^2\delta^2} k\delta(1 - x) \]
\[ \leq \frac{M}{2k\delta^2}. \]

Hence, for \( k \geq \frac{M}{\epsilon\delta^2} \)
\[ |f(x) - B_k(x)| = |\sum_{\{p\|\frac{p}{k} - x\| < \delta\}} (f(x) - f\left(\frac{p}{k}\right))r_p(x) + \sum_{\{p\|\frac{p}{k} - x\| \geq \delta\}} (f(x) - f\left(\frac{p}{k}\right))r_p(x)| \]
\[ \leq |\sum_{\{p\|\frac{p}{k} - x\| < \delta\}} (f(x) - f\left(\frac{p}{k}\right))r_p(x)| + |\sum_{\{p\|\frac{p}{k} - x\| \geq \delta\}} (f(x) - f\left(\frac{p}{k}\right))r_p(x)| \]
\[ \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \]

In conclusion,
\[ \|f(x) - B_k(x)\|_\infty = \sup_{|x| \leq 1} |f(x) - B_k(x)| < \epsilon. \]

It is essential for the conclusion that \([a, b]\) is a finite and closed interval. Theorem 1.38 will be fail if \([a, b]\) replaced by an open interval or unbounded interval.
The polynomials \( \{1, x, x^2, \ldots\} = \{x^k\}_{k=1}^\infty \) are linearly independent and do not span a finite-dimensional subspace of \( C[a, b] \). Indeed, if
\[
a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n = 0, \quad \forall x \in [a, b]
\]
then \( a_k \) must be zero for all \( k = 0, n \).

However, for a given \( n \in \mathbb{N} \), the vector space \( V := \text{span}\{1, x, x^2, \ldots, x^n\} \) is a finite-dimensional subspace of \( C[a, b] \) with the polynomial \( \{1, x, x^2, \ldots, x^n\} \) as a basis.

From this Theorem, we know that an element in infinite-dimensional vector space can be approximated by an element in finite-dimensional vector space. In classical Fourier analysis one expands functions in \( L^2(0, 1) \) in terms of complex exponential functions \( \{e^{2\pi ikx}\}_{k \in \mathbb{Z}} \). Let us for the moment consider a finite collection of exponential functions \( \{e^{i\lambda_k x}\}_{k=1}^n \), where \( \{\lambda_k\}_{k=1}^n \) is a sequence of real numbers. Unless \( \{\lambda_k\}_{k=1}^n \) contains repetitions, such a family of exponentials is always linearly independent:

**Lemma 1.39 [1]** Let \( \{\lambda_k\}_{k=1}^n \) be a sequence of real numbers, and assume that \( \lambda_k \neq \lambda_j \) for \( k \neq j \). Let \( I \subseteq \mathbb{R} \) be an arbitrary non-empty interval, and consider the complex exponentials \( \{e^{i\lambda_k x}\}_{k=1}^n \) as functions on \( I \). Then the functions \( \{e^{i\lambda_k x}\}_{k=1}^n \) are linearly independent.

**Proof** It is enough to prove that the functions \( \{e^{i\lambda_k x}\}_{k \in \mathbb{Z}} \) are linearly independent as function on any bounded interval \( [a, b] \) where \( a, b \in \mathbb{R}, a < b \).

Assume that
\[
\sum_{k=1}^n c_k e^{i\lambda_k x} = 0 \text{ for any } x \in [a, b].
\]

When \( x \) run through the interval \( \frac{a-b}{2}, \frac{b-a}{2} \), the variable \( x + \frac{a+b}{2} \) runs through \( [a, b] \). Therefore,
\[
\sum_{k=1}^n c_k e^{i\lambda_k (x + \frac{a+b}{2})} = 0 \text{ for any } x \in \frac{a-b}{2}, \frac{b-a}{2}.
\]

Taking the derivative \( j \) times, we get
\[
\sum_{k=1}^n d_k (i\lambda_k)^j e^{i\lambda_k x} = 0 \text{ for any } x \in \frac{a-b}{2}, \frac{b-a}{2}.
\]
where \( d_k := c_k e^{i \lambda_k x} \).

Letting \( x = 0 \) then

\[
\begin{pmatrix}
1 & 1 & \cdots & 1 \\
\lambda_1^1 & \lambda_2^1 & \cdots & \lambda_n^1 \\
\vdots & \vdots & \cdots & \vdots \\
\lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1}
\end{pmatrix}
\begin{pmatrix}
d_1 \\
d_2 \\
\vdots \\
d_n
\end{pmatrix}
= \begin{pmatrix}
0 \\
0 \\
\vdots \\
0
\end{pmatrix}.
\]

Besides

\[
A = \begin{pmatrix}
1 & 1 & \cdots & 1 \\
\lambda_1^1 & \lambda_2^1 & \cdots & \lambda_n^1 \\
\vdots & \vdots & \cdots & \vdots \\
\lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1}
\end{pmatrix}
\]

is Vandermonde matrix with determinant

\[
\det A = \prod_{i>j} (\lambda_i - \lambda_j) \neq 0.
\]

So, \( d_1 = d_2 = \ldots = d_n = 0 \).

Thus, \( c_1 = c_2 = \ldots = c_n = 0 \).

Hence, \( \{ e^{i \lambda_k x} \}_{k \in \mathbb{Z}} \) are linearly independent.

In words, Lemma 1.39 means that complex exponentials do not give natural examples of frame in finite-dimensional spaces: if \( \lambda_k \neq \lambda_j \) for \( k \neq j \), then the complex exponentials \( \{ e^{i \lambda_k x} \}_{k=1}^n \) form a basis for their span in \( L^2(I) \) for any interval \( I \) of finite length, and not overcomplete system. We can not obtain overcompleteness by adding extra exponentials (except by repeating some of the \( \lambda \)-values)-this will just enlarge the space.

Specially, as an important case we now consider the case where \( \lambda_k = 2 \). A function \( f \) which is a finite combination of the type

\[
f(x) = \sum_{k=N_1}^{N_2} c_k e^{2 \pi i k x} \quad \text{for some} \quad c_k \in \mathbb{C}; N_1, N_2 \in \mathbb{Z}; N_2 \geq N_1
\]  

(1.44)

is called a trigonometric polynomial. Trigonometric polynomial correspond to partial sums in Fourier series. A trigonometric polynomial \( f \) can also be written as a linear combination of functions \( \sin(2\pi x), \cos(2\pi x) \), in general with complex coefficients. It will be useful later to note that if \( f \) is real-valued and coefficients \( c_k \) in (1.44) are real, then \( f \) is linear combination of functions \( \cos(2\pi x) \) alone:
Lemma 1.40 [1] Assume that the trigonometric polynomial in (1.44) is real-valued and that $c_k \in \mathbb{R}$, then

$$f(x) = \sum_{k=N_1}^{N_2} c_k \cos(2\pi kx). \quad (1.45)$$

PROOF We have:

$$f(x) = \sum_{k=N_1}^{N_2} c_k e^{2\pi ikx}$$

$$= \sum_{k=N_1}^{N_2} c_k (\cos 2\pi kx + i \sin 2\pi kx).$$

Since $f$ is real-valued then

$$f(x) = \sum_{k=N_1}^{N_2} c_k \cos(2\pi kx). \quad \blacksquare$$

Note that we need the assumption that $c_k \in \mathbb{R}$: for example, the function

$$f(x) = \frac{1}{2i} e^{\pi ix} - \frac{1}{2i} e^{-\pi ix} = \sin x,$$

is real-valued, but does not have the form (1.45). Moreover, we mention that a positive-valued trigonometric polynomial has a square root, which is again a trigonometric polynomial:

Lemma 1.41 [1] Let $f$ be a positive-valued trigonometric polynomial of the form

$$f(x) = \sum_{k=N_1}^{N_2} c_k \cos(2\pi kx), \ c_k \in \mathbb{R}.$$  

Then there exists a trigonometric polynomial

$$g(x) = \sum_{k=0}^{N} d_k e^{2\pi ikx} \text{ with } d_k \in \mathbb{R}, \quad (1.46)$$

such that

$$|g(x)|^2 = f(x), \ \forall x \in \mathbb{R}. \quad (1.47)$$

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We can write
\[ f(x) = \sum_{m=0}^{M} a_m \cos(mx) \]
with \( a_m \in \mathbb{R} \).

Moreover, we have
\[ \cos(nx) = 2 \cos((n-1)x) \cos x + \cos((n-2)x), \quad \forall n \geq 2, \ n \in \mathbb{R}. \]  

(1.48)

So, \( \cos(mx) \) can be rewritten as a polynomial of degree \( m \) of \( \cos x \).

Therefore, \( f(x) = p_f(\cos x) \), where \( p_f \) is a polynomial of degree \( M \) with real coefficients. This polynomial can be factored,
\[ p_f(c) = \alpha M \prod_{j=1}^{M} (c - c_j), \]
where the zeros \( c_j \) of \( p_f \) appear either in complex duplets \( c_j, \bar{c}_j \), or in real singlets.

We can also write
\[ f(x) = e^{iMx} P_f(e^{-ix}), \]  

(1.49)

where \( P_f \) is a polynomial of degree \( 2M \). For \( |z| = 1 \), we have
\[ P_f(z) = z^M \alpha \prod_{j=1}^{M} (z^{1/2} - c_j) = \alpha \prod_{j=1}^{M} (\frac{1}{2} - c_j z + \frac{1}{2} z^2). \]  

(1.50)

Indeed, we have
\[ P_f(e^{-ix}) = e^{-iMx} \alpha \prod_{j=1}^{M} (\frac{e^{-ix} + e^{ix}}{2} - c_j). \]

Therefore,
\[ e^{iMx} P_f(e^{-ix}) = \alpha \prod_{j=1}^{M} (\frac{e^{-ix} + e^{ix}}{2} - c_j) \]
\[ = \alpha \prod_{j=1}^{M} (\cos x - c_j) \]
\[ = p_f(\cos x) = f(x). \]

If \( c_j \) is real, then the zeros of \( \frac{1}{2} - c_j z + \frac{1}{2} z^2 \) are \( c_j \pm \sqrt{c_j^2 - 1} \).

For \( |c_j| \geq 1 \), these are two real zeros (degenerate if \( c_j = \pm 1 \)) of the form \( r_j, r_j^{-1} \).

For \( |c_j| < 1 \), the two zeros are complex conjugate and of absolute value 1, i.e., they are of the form \( e^{i\alpha_j}, e^{-i\alpha_j} \). Then \( \alpha_j, -\alpha_j \) be the zeros of \( f(x) \), or
\[ f(x) = ((x - \alpha_j)(x + \alpha_j))^n Q(x). \]  

(1.51)
In order not to cause any contradiction with \( f(x) \geq 0 \), these zeros must be have even multiplicity.
If \( c_j \) is not real, then we consider it together with \( c_k = c_{j}^{-1} \). The polynomial 
\[
\left( \frac{1}{2} - c_j z + \frac{1}{2} z^2 \right) \left( \frac{1}{2} - \overline{c_j} z + \frac{1}{2} z^2 \right)
\]
has four zeros, \( c_j \pm \sqrt{c_j^2 - 1} \) and \( \overline{c_j} \pm \sqrt{\overline{c_j}^2 - 1} \). The four zeros are all different, and form a quadruplet \( z_j, \, z_j^{-1}, \, \overline{z_j}, \, \overline{z_j}^{-1} \).
Therefore, we have
\[
P_f(z) = \frac{1}{2} \left[ \prod_{j=1}^{J} (z - z_j)(z - z_j^{-1})(z - \overline{z_j})(z - \overline{z_j}^{-1}) \right] \times \\
\times \prod_{k=1}^{K} (z - e^{\text{i} \alpha_k})^2(z - e^{-\text{i} \alpha_k})^2 \times \prod_{\ell=1}^{L} (z - r_\ell)(z - r_\ell^{-1}),
\]
where we have regrouped the three different kind of zeros.
For \( z = e^{-\text{i} x} \) on the unit circle, we have
\[
|e^{-\text{i} x} - z_0)(e^{-\text{i} x} - \overline{z_0}^{-1})| = |e^{-\text{i} x} - z_0)(\frac{1}{e^{\text{i} x}} - \frac{1}{\overline{z_0}})| \\
= |z_0|^{-1}|e^{-\text{i} x} - z_0)(e^{\text{i} x} - \overline{z_0})| \\
= |z_0|^{-1}|e^{-\text{i} x} - z_0)|^2.
\]
Consequently,
\[
f(x) = |f(x)| = |P_f(e^{-\text{i} x})| \\
= \frac{1}{2} |a_M| \prod_{j=1}^{J} |z_j|^{-2} \prod_{\ell=1}^{L} |z_\ell^{-1}| \times \prod_{j=1}^{J} (e^{-\text{i} x} - z_j)(e^{-\text{i} x} - \overline{z_j})|^2 \times \\
\times \prod_{k=1}^{K} (e^{-\text{i} x} - e^{\text{i} \alpha_k})(e^{-\text{i} x} - e^{-\text{i} \alpha_k})^2 \times \prod_{\ell=1}^{L} (e^{-\text{i} x} - r_\ell)^2 \\
= |g(x)|^2,
\]
where
\[
g(x) = \frac{1}{2} |a_M| \prod_{j=1}^{J} |z_j|^{-2} \prod_{\ell=1}^{L} |z_\ell^{-1}| \frac{1}{2} \prod_{j=1}^{J} (e^{-\text{i} x} - z_j)(e^{-\text{i} x} - \overline{z_j}) \times \\
\times \prod_{k=1}^{K} (e^{-\text{i} x} - e^{\text{i} \alpha_k})(e^{-\text{i} x} - e^{-\text{i} \alpha_k}) \times \prod_{\ell=1}^{L} (e^{-\text{i} x} - r_\ell)
\]
\[ = \left| \frac{1}{2} a_M \prod_{j=1}^{J} |z_j|^{-2} \prod_{\ell=1}^{L} |z_\ell^{-1}|^{\frac{1}{2}} \right| \times \prod_{j=1}^{J} (e^{-2ix} - 2e^{-ix} \text{Re}z_j + |z_j|^2) \times \prod_{k=1}^{K} (e^{-2ix} - 2e^{-ix} \cos \alpha_k + 1) \times \prod_{\ell=1}^{L} (e^{-ix} - r_\ell) \]

is clear a trigonometric polynomial of order $M$ with real coefficients.

**Example 1.42**

Consider $f(x) = 1 + \cos 2x$.

We find $g(x) = d_0 + d_1 e^{ix} + d_2 e^{2ix}$ satisfies $|g(x)|^2 = f(x)$. We have

\[
|g(x)|^2 = |d_0 + d_1 \cos x + d_2 \cos 2x + i(d_1 \sin x + d_2 \sin 2x)|^2
= (d_0 + d_1 \cos x + d_2 \cos 2x)^2 + (d_1 \sin x + d_2 \sin 2x)^2
= d_0^2 + d_1^2 + d_2^2 + 2(d_0d_1 + d_1d_2) \cos x + 2d_0d_2 \cos 2x = 1 + \cos 2x.
\]

Therefore,

\[
\begin{align*}
\left\{ \\
\frac{d_0^2 + d_1^2 + d_2^2}{2(d_0d_1 + d_1d_2)} = 0 \\
2d_0d_2 = 1
\end{align*}
\]

Thus,

\[
\begin{align*}
\left\{ \\
d_1 = 0 \\
d_0 = d_2 = \pm \frac{1}{\sqrt{2}}
\end{align*}
\]

Hence,

\[ g(x) = \pm \frac{1}{\sqrt{2}} (1 + e^{2ix}). \]

**Remark 1.43**

1. This proof is constructive. It uses factorization of a polynomial of degree $M$, however, which has to be done numerically and may lead to problems if $M$ is large and some zeros are closed together. Note that in this proof we need to factor a polynomial of degree only $M$, unlike some procedures which factor directly $P_f$, a polynomial of degree $2M$. 

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2. This procedure of "extracting the square root" is also called \textit{spectral factorization} in the engineering literature.

3. The polynomial \( g(x) \) is not unique! For \( M \) odd, for instance, \( P_f \) may have quadruplets of complex zeros, and 1 pair of real zeros. In each quadruplet we can choose either \( z_j, z_j^{-1} \) to make up \( g(x) \), or \( z_j^{-1}, z_j^{-2} \); for each duplet we can choose either \( r_\ell \) or \( r_\ell^{-1} \). This makes already for \( 2^{M+1} \) different choices for \( g(x) \). Moreover, we can always multiply \( g(x) \) with \( e^{inx} \), \( n \) arbitrary in \( \mathbb{Z} \).

\textbf{Example 1.44}

Consider \( f(x) = 1 + \cos x \).

We calculate directly \( g(x) = d_0 + d_1 e^{ix} \) for which \( |g(x)|^2 = f(x) \).

Then we obtain \( g(x) = \pm \frac{1}{\sqrt{2}} (1 + e^{ix}) \).

Moreover, we can choose \( g'(x) = \pm \frac{1}{\sqrt{2}} (e^{ix} + e^{i2x}) \) also satisfy \( |g'(x)|^2 = f(x) \).

Note that by definition, the function \( g(x) \) in (1.46) is complex-valued, unless \( f \) is constant. Actually, despite the fact that \( f \) is assumed to be positive, there might not exist a \textit{positive} trigonometric polynomial \( g \) satisfying (1.47).
The thesis can be fully understood with an elementary knowledge of linear algebra. This presents basic results in finite-dimensional vector spaces with an inner product.

Section 1.1 contains the basic properties of frames. For example, it is proved that every set of vectors \( \{f_k\}_{k=1}^m \) in a vector space with an inner product is a frame for \( \text{span}\{f_k\}_{k=1}^m \). We have proved the existence of coefficients minimizing the \( \ell^2 \)-norm of the coefficients in a frame expansion and showed how a frame for a subspace lead to a formula for the orthogonal projection onto the subspace.

In Section 1.2 and Section 1.3, we have considered frames on \( \mathbb{C}^n \); in particular, we’ve proved how we can obtain an overcomplete frame by a projection of a basis for a larger space. We also proved that the vectors \( \{f_k\}_{k=1}^m \) is a frame for \( \mathbb{C}^n \) can be considered as the first \( n \) coordinates of some vectors in \( \mathbb{C}^m \) constituting a basis for \( \mathbb{C}^m \), and that the frame property for \( \{f_k\}_{k=1}^m \) is equivalent to certain properties for the \( m \times n \) matrix having the vectors \( f_k \) as rows.

In Section 1.4 we have proved that the canonical coefficients from the frame expansion arise naturally by considering the pseudo-inverse of the pre-frame operator, and we showed how to find it in term of the singular value decomposition. Finally, in Section 1.5 we connected frames in finite-dimensional vector spaces with the infinite-dimensional constructions which we expect study latter.
References

