THE PAPER

“SOME ELEMENTARY INEQUALITIES”

OF G. BENNETT

Undergraduate Thesis
Advanced Undergraduate Program in Mathematics

Thesis advisor: Dr. Dang Anh Tuan

Hanoi - 2013
Acknowledgments

I would like to express my sincere thank to my thesis advisor Dr. Dang Anh Tuan for the help, support, patience in writing of this thesis.

I would like to show my gratitude to other teachers at the Mathematics Department of University of Science for their teaching. I also want to thank my friends in K54-Advanced Math.

Last, I especially thank to my parents for their care, encouragement, and supports.
## Contents

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Preface</td>
<td>4</td>
</tr>
<tr>
<td><strong>1</strong> Some Inequalities</td>
<td>5</td>
</tr>
<tr>
<td>1.1 Notations and some basis inequalities</td>
<td>5</td>
</tr>
<tr>
<td>1.2 Hardy’s extension inequalities</td>
<td>7</td>
</tr>
<tr>
<td><strong>2</strong> Factorable matrices</td>
<td>17</td>
</tr>
<tr>
<td>2.1 Factorable matrices</td>
<td>17</td>
</tr>
<tr>
<td>2.2 Some Corollaries of Theorem 2.1.1</td>
<td>19</td>
</tr>
<tr>
<td>2.3 Example</td>
<td>26</td>
</tr>
<tr>
<td><strong>3</strong> Weighted mean matrices</td>
<td>33</td>
</tr>
<tr>
<td>3.1 Weighted mean matrices</td>
<td>33</td>
</tr>
<tr>
<td>3.2 Some Corollaries of Theorem 3.1.1</td>
<td>43</td>
</tr>
<tr>
<td><strong>4</strong> Littlewood’s problem</td>
<td>51</td>
</tr>
<tr>
<td>4.1 Littlewood’s problem</td>
<td>51</td>
</tr>
<tr>
<td>4.2 Reverse Littlewood’s inequality</td>
<td>54</td>
</tr>
<tr>
<td>Conclusion</td>
<td>61</td>
</tr>
<tr>
<td>Bibliography</td>
<td>61</td>
</tr>
</tbody>
</table>
Preface

The aim of this thesis presents extensive Hardy’s inequalities and some results involves it. Moreover, the thesis gives some conditions such that some special matrices are bounded operators from $l^p$ into $l^q$ and we consider Littlewood’s problem. The thesis are divided into four chapters:

- Chapter 1 gives some notations and some basis inequalities which will be use to prove some Theorems and Corollaries in the thesis. Besides, we give two extensions of Hardy’s inequalities.

- In Chapter 2, we consider the factorable matrix and give some conditions to the factorable matrix is bounded operator from $l^p$ into $l^q$. We also give many interesting Corollaries of it. After that, we give an example of factorable matrix.

- Next, we consider the weighted mean matrix and some results involve it in Chapter 3. On other hand, we consider the important Theorem in the thesis of J. Cartlidge.

- Finally, in Chapter 4 we consider Littlewood’s problem and we will determine the constant $K$ of Littlewood’s problem in special case and general case. The reverse Littlewood’s inequality is also considered.

The main materials of the thesis were taken from the paper of G. Bennett [1]. We have also borrowed extensively from the book of G. H. Hardy, J. E. Littlewood and G. Pólya [4]. We also use the papers of D. Sylvain [5] and G. Bennett, K-G. Grosse-Erdmann [2]. A note of P. Gao [3] is also used in this thesis.
Chapter 1

Some Inequalities

1.1 Notations and some basis inequalities

We shall be concerned with matrix transformations of $l^p$ spaces. Here as usual, $l^p$ denotes the space of real-valued sequences $x$ satisfying $\|x\|_p \leq \infty$ where

$$\|x\|_p = \sum_{k=1}^{\infty} |x_k|^p$$

and

$$\|x\|_{\infty} = \sup_{k \in \mathbb{N}} |x_k|.$$ 

For $1 \leq p < \infty$ we denote the conjugate exponent by $p^*$ so that $p^* = p/(p-1)$. The norm of matrix $A$, mapping $l^p$ into $l^q$ is given by

$$\|A\|_{p,q} = \sup \{\|Ax\|_q : \|x\|_p \leq 1\},$$

where

$$\|Ax\|_q = \left( \sum_{n=1}^{\infty} \left| \sum_{k=1}^{n} a_{nk} x_k \right|^q \right)^{1/q}.$$ 

After that, we gives some basis inequalities.

**Theorem 1.1.1** (Holder’s inequality). Let $a_i, b_i \geq 0$ and $\frac{1}{p} + \frac{1}{q} = 1$ for $p, q > 1$ then

$$\sum_{i=1}^{\infty} a_i b_i \leq \left( \sum_{i=1}^{\infty} a_i^p \right)^{1/p} \left( \sum_{i=1}^{\infty} b_i^q \right)^{1/q}.$$  \hspace{1cm} (1.1.1)

Moreover, one has equality only when the sequences $(a_1^p, \ldots, a_i^p, \ldots)$ and $(b_1^q, \ldots, b_i^q, \ldots)$ are proportional, i.e., $a_i = Cb_i^{p-1}$.

**Proof.** It is classical inequality. \hfill $\square$

Next, in two following Lemmas, we assume throughout that $(a_n)$ and $(b_n)$ are sequences of non-negative terms. We consider $X$ and $Y$ are the sum of two non-negative series, we call $X$ and $Y$ are equivalent iff

$$K_1 X \leq Y \leq K_2 X,$$

where $K_1, K_2$ are positive constants and we denote $X \sim Y$. 

5
Lemma 1.1.1 (Power Rule). If $p \geq 1$, then, for all $n \in \mathbb{N}$

$$
\sum_{k=1}^{n} a_k \left( \sum_{j=1}^{k} a_j \right)^{p-1} \leq \left( \sum_{k=1}^{n} a_k \right)^p \leq p \sum_{k=1}^{n} a_k \left( \sum_{j=1}^{k} a_j \right)^{p-1}.
$$

(1.1.2)

The inequalities reverse direction if $0 < p < 1$ (and $a_1 > 0$).

Proof. Case 1: $p \geq 1$

First, we will prove the left-hand inequality. Because $k \leq n$ and $p \geq 1$ so

$$
\left( \sum_{j=1}^{k} a_j \right)^{p-1} \geq \left( \sum_{j=1}^{n} a_j \right)^{p-1}.
$$

(1.1.3)

Then

$$
\left( \sum_{k=1}^{n} a_k \right)^p = \sum_{k=1}^{n} a_k \left( \sum_{k=1}^{n} a_k \right)^{p-1} \leq \sum_{k=1}^{n} a_k \left( \sum_{j=1}^{n} a_j \right)^{p-1} \geq \sum_{k=1}^{n} a_k \left( \sum_{j=1}^{k} a_j \right)^{p-1}.
$$

Second, we prove that the right-hand inequality. Let $A_k = \sum_{j=1}^{k} a_j$. Because $p \geq 1$, then

$$
\int_{A_{k-1}}^{A_k} x^{p-1} dx \leq (A_k - A_{k-1}) A_k^{p-1} = a_k A_k^{p-1}.
$$

(1.1.4)

Thus,

$$
A_k^p - A_{k-1}^p = p \int_{A_{k-1}}^{A_k} x^{p-1} dx \leq p a_k A_k^{p-1}.
$$

Take summation on $k$ we get

$$
\sum_{k=1}^{n} (A_k^p - A_{k-1}^p) = A_n^p \leq p \sum_{k=1}^{n} a_k \left( \sum_{j=1}^{k} a_j \right)^{p-1}.
$$

Hence, we get the right-hand inequality.

The same argument applies when $0 < p < 1$, which case the inequality (1.1.3) and (1.1.4) are reversed and we obtain the inverse inequalities. □

Lemma 1.1.2 (Power Rule for Tails). If $p \geq 1$, then

$$
\sum_{k=n}^{\infty} a_k \left( \sum_{j=k}^{\infty} a_j \right)^{p-1} \leq \left( \sum_{k=n}^{\infty} a_k \right)^p \leq p \sum_{k=n}^{\infty} a_k \left( \sum_{j=k}^{\infty} a_j \right)^{p-1}.
$$

(1.1.5)

The inequality reverse direction when $0 < p < 1$ if the term of $a$ are positive.
CHAPTER 1. SOME INEQUALITIES

Proof. This may be proved directly an argument similarly to that used in Lemma 1.1.1. Case 1: \( p \geq 1 \), for left-hand inequality, because \( k \geq n \) and \( p \geq 1 \) so

\[
\left( \sum_{j=k}^{\infty} a_j \right)^{p-1} \leq \left( \sum_{j=n}^{\infty} a_j \right)^{p-1}.
\]  

(1.1.6)

Then

\[
\sum_{k=n}^{\infty} a_k \left( \sum_{j=k}^{\infty} a_j \right)^{p-1} \leq \sum_{k=n}^{\infty} a_k \left( \sum_{j=n}^{\infty} a_j \right)^{p-1} = \sum_{k=n}^{\infty} a_k \left( \sum_{k=n}^{\infty} a_k \right)^{p-1} = \left( \sum_{k=n}^{\infty} a_k \right)^p.
\]

For right-hand inequality: We also let \( A_k = \sum_{j=k}^{\infty} a_j \). Since \( p \geq 1 \) so that we have

\[
A_k^p - A_{k+1}^p = p \int_{A_{k+1}}^{A_k} x^{p-1} \, dx \leq p(A_k - A_{k+1})A_k^{p-1} = p a_k A_k^{p-1}.
\]  

(1.1.7)

So

\[
\sum_{k=n}^{\infty} \left( A_k^p - A_{k+1}^p \right) = A_n^p \leq p \sum_{k=n}^{\infty} a_k \left( \sum_{j=k}^{\infty} a_j \right)^{p-1}.
\]

Similarly, when \( 0 < p < 1 \) the inequalities (1.1.6) and (1.1.7) are reversed. We complete the Lemma Power Rule for Tails.

\[\square\]

1.2 Hardy’s extension inequalities

Theorem 1.2.1 (Hardy’s extension inequality I). Let \( p > 1, \lambda_n > 0, a_n \geq 0, \) and

\[\Lambda_n = \sum_{k=1}^{n} \lambda_k, \ A_n = \sum_{k=1}^{n} \lambda_k a_k, \]

then

\[
\sum_{n=1}^{\infty} \lambda_n \left( \frac{A_n}{\Lambda_n} \right)^p \leq \left( \frac{p}{p-1} \right)^p \sum_{n=1}^{\infty} \lambda_n a_n^p.
\]  

(1.2.1)
CHAPTER 1. SOME INEQUALITIES

Proof. We write \( \alpha_n = A_n / \Lambda_n \). We consider

\[
\lambda_n \alpha_n^p - \frac{p}{p-1} \lambda_n a_n \alpha_n^{p-1} = \lambda_n \alpha_n^p - \frac{p}{p-1} \alpha_{n-1}^{p-1} \left[ \Lambda_n \alpha_n - \Lambda_{n-1} \alpha_{n-1} \right]
\]

\[
= \left( \lambda_n - \Lambda_n \frac{p}{p-1} \right) \alpha_n^p + \frac{p}{p-1} \Lambda_{n-1} \alpha_{n-1}^{p-1}
\]

\[
\leq \left( \lambda_n - \Lambda_n \frac{p}{p-1} \right) \alpha_n^p + \frac{p \Lambda_{n-1}}{p-1} \left[ \frac{\alpha_{n-1}^p}{p} + \frac{\alpha_n^{p-1}}{q} \right]
\]

\[
= \left( \lambda_n - \Lambda_n \frac{p}{p-1} \right) \alpha_n^p + \frac{\Lambda_{n-1}}{p-1} \left[ \alpha_{n-1}^p + (p-1) \alpha_n^p \right]
\]

\[
= \frac{1}{p-1} \left[ (p \lambda_n - \lambda_n - p(\Lambda_n - \Lambda_{n-1}) - \Lambda_{n-1} \alpha_n^p + \Lambda_{n-1} \alpha_{n-1}^p) \right]
\]

\[
= \frac{1}{p-1} \left[ (p \lambda_n - \lambda_n - p \lambda_n - \Lambda_{n-1} \alpha_n^p + \Lambda_{n-1} \alpha_{n-1}^p) \right]
\]

\[
= \frac{1}{p-1} \left[ -\Lambda_n \alpha_n^p + \Lambda_{n-1} \alpha_{n-1}^p \right].
\]

Hence,

\[
\sum_{n=1}^{N} \lambda_n \alpha_n^p - \frac{p}{p-1} \sum_{n=1}^{N} \lambda_n a_n \alpha_n^{p-1} \leq \frac{1}{p-1} \sum_{n=1}^{N} \left[ \Lambda_{n-1} \alpha_{n-1}^p - \Lambda_n \alpha_n^p \right]
\]

\[
= \frac{1}{p-1} \left[ 0 - \Lambda_1 \alpha_1^p + \Lambda_1 \alpha_1^p - \Lambda_2 \alpha_2^p + \cdots - \Lambda_N \alpha_N^p \right]
\]

\[
= -\frac{1}{p-1} \Lambda_N \alpha_N^p \leq 0.
\]

Thus,

\[
\sum_{n=1}^{N} \lambda_n \alpha_n^p \leq \frac{p}{p-1} \sum_{n=1}^{N} \lambda_n a_n \alpha_n^{p-1}. \quad (1.2.2)
\]

Using Holder’s inequality with indexes \( p, p/(p-1) \) on the right-hand side of (1.2.2) we have

\[
\sum_{n=1}^{N} \lambda_n \alpha_n^p \leq \frac{p}{p-1} \left( \sum_{n=1}^{N} \lambda_n a_n^p \right)^{1/p} \left( \sum_{n=1}^{N} \lambda_n \alpha_n^p \right)^{(p-1)/p}
\]

Dividing the above inequality by the last factor on the right-hand side and raising the result to the \( p \)th power, we obtain

\[
\sum_{n=1}^{N} \lambda_n \alpha_n^p \leq \left( \frac{p}{p-1} \right)^p \sum_{n=1}^{N} \lambda_n a_n^p \quad (1.2.3)
\]

When we make \( N \) tend to infinity we obtain (1.2.1). \( \square \)

In the above Theorem, let \( \lambda_k = 1 \), we get the Hardy’s inequality

\[
\sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{k=1}^{n} a_k \right)^p \leq \left( \frac{p}{p-1} \right)^p \sum_{n=1}^{\infty} a_n^p.
\]
Theorem 1.2.2 (Hardy’s extension inequality II). Let \( r > s \geq 1 \) and \( u, v, w \) be \( N \)-tuples with non-negative entries. If
\[
\sum_{n=1}^{m} u_n \left( \sum_{k=1}^{n} v_k \right)^r \leq \left( \sum_{k=1}^{m} v_k \right)^s \quad \text{for} \quad m = 1, 2, \ldots, N \tag{1.2.4}
\]
then
\[
\sum_{n=1}^{N} u_n \left( \sum_{k=1}^{n} v_k w_k \right)^r \leq K(r, s) \left( \sum_{k=1}^{N} v_k w_k^{r/s} \right)^s \tag{1.2.5}
\]
where
\[
K(r, s) = \left( \frac{r}{r - s} \right)^r. \tag{1.2.6}
\]

In the above Theorem, we can assume that \( u_N > 0 \). Indeed, if \( u_N = 0 \), inequality (1.2.4) becomes
\[
\sum_{n=1}^{m} u_n \left( \sum_{k=1}^{n} v_k \right)^r \leq m \sum_{k=1}^{m} v_k \quad \text{for} \quad m = 1, \ldots, N-1.
\]
And (1.2.5) becomes
\[
\sum_{n=1}^{N-1} u_n \left( \sum_{k=1}^{n} v_k w_k \right)^r \leq K(r, s) \left( \sum_{k=1}^{N-1} v_k w_k^{r/s} \right)^s.
\]

Thus, we obtain an equivalent with \((N - 1)\)-tuples. To prove Theorem 1.2.2 we need to prove some following Lemmas.

**Lemma 1.2.1.** Fix \( N \) and non-negative \( N \)-tuples \( u, w \) we say that \( w \) is decreasing if \( w_i \geq w_j \) whenever \( i < j \) both \( v_i > 0, v_j > 0 \). If Theorem 1.2.2 holds for decreasing above \( w \) then it is true for arbitrary \( w \).

**Lemma 1.2.2.** If Theorem 1.2.2 is true for \( s = 1 \), it holds for \( s > 1 \).

**Lemma 1.2.3.** Let \( r > 1, s = 1 \) and \( u, v, w \) be \( N \)-tuples with non-negative entries. In addition, the decreasing \( w \) is defined in Lemma 1.2.1 that is satisfied
\[
\sum_{n=1}^{m} u_n \left( \sum_{k=1}^{n} v_k \right)^r \leq \sum_{k=1}^{m} v_k \quad \text{for} \quad m = 1, \ldots, N. \tag{1.2.7}
\]

Then
\[
\sum_{n=1}^{N} u_n \left( \sum_{k=1}^{n} v_k w_k \right)^r \leq \left( \frac{r}{r - 1} \right)^r \sum_{k=1}^{N} v_k w_k^r, \tag{1.2.8}
\]

Now, we will prove the above Lemmas.

**Proof of Lemma 1.2.1.** If \( w \) is not decreasing that means there exists integers \( i, j \) with \( 1 \leq i < j < N \) such that \( w_i < w_j \) both \( v_i > 0, v_j > 0 \). We have \( w = (w_1, w_2, \ldots, w_N) \) with \( w_i \geq 0 \) for all \( i \). Take a new \( N \)-tuples, \( w' \), as follow
\[
w' = (w_1, w_2, \ldots, w_{i-1}, w'_i, w_{i+1}, \ldots, w_{j-1}, w'_j, w_{j+1}, \ldots, w_n),
\]
where
\[ w_i^{r/s} = w_j^{r/s} = \frac{v_i w_i^{r/s} + v_j w_j^{r/s}}{v_i + v_j}. \]  

We claim that
\[ \sum_{k=1}^{N} v_k w_k^{r/s} = \sum_{k=1}^{N} v_k (w_k')^{r/s}. \]  

Indeed,
\[ \sum_{k=1}^{N} v_k w_k^{r/s} = v_1^{r/s} + \cdots + v_i^{r/s} + v_j^{r/s} + \cdots + v_N^{r/s}. \]
\[ = v_1^{r/s} + \cdots + \frac{v_i^{r/s} + v_j^{r/s}}{v_i + v_j} + \cdots + \frac{v_i^{r/s} + v_j^{r/s}}{v_i + v_j} + \cdots + v_N^{r/s}. \]
\[ = \sum_{k=1}^{N} v_k w_k^{r/s}. \]

By (1.2.10), we have the right side of (1.2.5) is unchanged when \( w \) is replaced by \( w' \). Now, we will show that the left side of (1.2.5) increases strictly when \( w \) is replaced by \( w' \). From (4.1.1), \( r > s \geq 1 \) and by assuming of the hypothesis we have
\[ w_i' = w_j' = \left( \frac{v_i^{r/s} + v_j^{r/s}}{v_i + v_j} \right)^{s/r} \]
\[ > \left( \frac{v_i^{r/s} + v_j^{r/s}}{v_i + v_j} \right)^{s/r} \]
\[ = w_i. \]  

Similarly, we also have
\[ w_i' = w_j' = \left( \frac{v_i^{r/s} + v_j^{r/s}}{v_i + v_j} \right)^{s/r} \]
\[ < \left( \frac{v_i^{r/s} + v_j^{r/s}}{v_i + v_j} \right)^{s/r} \]
\[ = w_j. \]  

We will compare
\[ \sum_{k=1}^{n} v_k w_k \] and \[ \sum_{k=1}^{n} v_k w'_k. \]

For \( n < i \), we obtain
\[ \sum_{k=1}^{n} v_k w_k = \sum_{k=1}^{n} v_k w'_k. \]

For \( i \leq n < j \), we have
\[ \sum_{k=1}^{n} v_k w_k = v_1 w_1 + \cdots + v_i w_i + \cdots + v_n w_n \]
\[ < v_1 w_1 + \cdots + v_i w_i' + \cdots + v_n w_n \]
\[ = \sum_{k=1}^{n} v_k w'_k. \]
For \( n \geq j \), we apply the Jensen’s inequality for convex function \( f(x) = x^{r/s} \)

\[
\frac{v_i}{v_i + v_j} w_i^{r/s} + \frac{v_j}{v_i + v_j} w_j^{r/s} \geq \left( \frac{v_i}{v_i + v_j} w_i + \frac{v_j}{v_i + v_j} w_j \right)^{r/s}.
\]

So that

\[
(v_i + v_j)w'_i \geq (v_i + v_j)\frac{v_i}{v_i + v_j} w_i + \frac{v_j}{v_i + v_j} w_j = v_iw_i + v_jw_j.
\]

Hence,

\[
\sum_{k=1}^{n} v_kw_k \leq \sum_{k=1}^{n} v_kw'_k.
\]

Therefore, we always have for all \( n \in \mathbb{N} \)

\[
\sum_{k=1}^{n} v_kw_k \leq \sum_{k=1}^{n} v_kw'_k.
\]

We have shown that the left side of (1.2.5) is increasing strictly when \( w \) is replaced by \( w' \).

Next, we will point out the above process from arbitrary \( w \) to the decreasing \( w^* \) is defined in Lemma 1.2.1 is finite by induction. In the process, if there exists \( v_i = 0 \), the decreasing or nondecreasing of \( w \) is not influenced by \( w_i \). Thus, we can assume that \( v \) is positive \( N \)-tuples.

For \( N = 2 \), \( w = (w_1, w_2) \). If \( w_1 \geq w_2 \), we stop.

If \( w_1 < w_2 \), we change

\[
w' = (w'_1, w'_2) \quad \text{such that} \quad w_1 < w'_1 = w'_2 < w_2.
\]

Hence, we change once time.

We assume that \( w = (w_1, \ldots, w_{N-1}) \) and we need at most

\[
\frac{(N - 2)(N - 1)}{2}
\]

to give decreasing \((N - 1)\)-tuples \( w^* \).

We will show that it holds for \( n = N \), that means after at most

\[
\frac{N(N - 1)}{2}
\]

the process will be stops.

We consider \( w = (w_1, w_2, \ldots, w_N, w_{N-1}) \) which is satisfied

\[
w_1 \geq w_2 \geq \cdots \geq w_{N-1}.
\] (1.2.13)

- If \( w_{N-1} \geq w_N \), we will stop.

- If \( w_{N-1} < w_N \), we will compare \( w_1 \) and \( w_N \).

  If \( w_1 < w_N \), we’ll change \( w_1 \) and \( w_N \) to obtain

  \[
w' = (w'_1, w_2, \ldots, w_{N-1}, w'_N) \quad \text{such that} \quad w_2 \leq w_1 < w'_1 = w'_N.
\]
We continue to change $w_2$ and $w'_N$, we get

$$w'' = (w'_1, w'_2, w_3, \ldots, w_{N-1}, w''_N)$$

such that $w_3 \leq w_2 < w'_2 = w''_N < w'_1$.

So on the process and we need $N - 1$ times to give a new decreasing $N$-tuples $w^*$. If $w_1 \geq w_N$, we have $w_1 \geq w_N > w_{N-1}$ then there exists

$$j \in \{1, 2, \ldots, N - 2\}$$

such that $w_j \geq w_N \geq w_{j+1}$.

We also do similarly as the process $w_1 < w_N$ and after $N - j - 1$ steps, we get the a decreasing $N$-tuples $w^*$.

Therefore, we want to change from $w$ is satisfied (1.2.13) to decreasing $N$-tuples $w^*$ then we need at most $N - 1$ steps.

Combining the assumption, to change from arbitrary the $N$-tuples $w$ to the decreasing $N$-tuples $w^*$ we need at most

$$\frac{(N - 2)(N - 1)}{2} + N - 1 = \frac{N(N - 1)}{2}$$

steps.

We complete the proof of Lemma 1.2.1.

Proof of Lemma 1.2.2. Assuming that Theorem 1.2.2 holds for $s = 1$, that means if

$$\sum_{n=1}^{m} u_n \left( \sum_{k=1}^{n} v_k \right)^r \leq \sum_{k=1}^{m} v_k,$$  \hspace{1cm} (1.2.14)

then

$$\sum_{n=1}^{N} u_n \left( \sum_{k=1}^{n} v_kw_k \right)^r \leq K(r, 1) \sum_{k=1}^{N} v_kw_k^r,$$  \hspace{1cm} (1.2.15)

where

$$K(r, 1) = \left( \frac{r}{r-1} \right)^r.$$

For $s > 1$, let $x$ be the a non-negative $N$-tuple with

$$\|x\|_{s^*} = \left( \sum_{n=1}^{N} x^{s^*} \right)^{1/s^*} = 1,$$

where

$$s^* = \frac{s}{s-1}.$$

Assuming inequality (1.2.4) holds. Using Holder’s inequality and inequality (1.2.4), we get

$$\sum_{n=1}^{m} x_n^{1/s} u_n^{1/s} \left( \sum_{k=1}^{n} v_k \right)^{r/s} \leq \|x\|_{s^*} \left( \sum_{n=1}^{m} u_n \left( \sum_{k=1}^{n} v_k \right)^r \right)^{1/s} \leq \sum_{k=1}^{m} v_k.$$
We see that the above inequality is the inequality (1.2.14) with $u_n$ is replaced by $x_n u_n^{1/s}$, $r$ by $r/s$. Hence, the inequality (1.2.15) holds with $u_n$ is replaced by $x_n u_n^{1/s}$, $r$ by $r/s$

$$
\frac{N}{n=1} x_n u_n^{1/s} \left( \sum_{k=1}^{n} v_k w_k \right)^{r/s} \leq K(r/s, 1) \frac{N}{n=1} v_k w_k^{r/s}.
$$

Taking the supermum on the left over all $x$ satisfying $\|x\|_{s^*} = 1$ we get

$$
\sup_{\|x\|_{s^*}=1} \frac{N}{n=1} x_n u_n^{1/s} \left( \sum_{k=1}^{n} v_k w_k \right)^{r/s} \leq K(r/s, 1) \frac{N}{n=1} v_k w_k^{r/s}. \tag{1.2.16}
$$

Let $y = \left( u_n^{1/s} \left( \sum_{k=1}^{n} v_k w_k \right)^{r/s} \right)^{N}$ and $x = (x_n)_{n=1}^{N}$. We have

$$
\sup_{\|x\|_{s^*}=1} \frac{N}{n=1} x_n u_n^{1/s} \left( \sum_{k=1}^{n} v_k w_k \right)^{r/s} = \sup_{\|x\|_{s^*}=1} <y, x>.
$$

We claim that

$$
\sup_{\|x\|_{s^*}=1} |<y, x>| = \|y\|_s. \tag{1.2.17}
$$

By Holder’s inequality, we have

$$
|<y, x>| \leq \|y\|_s \|x\|_{s^*} = \|y\|_s. \tag{1.2.18}
$$

On the other hand, we choose

$$
|x_k| = \left( \frac{|y_k|^s}{\|y\|_s^s} \right)^{1/s^*}
$$

is satisfied $\|x\|_{s^*} = 1$. So that

$$
|<y, x>| = \sum_{k=1}^{\infty} |y_k||x_k| = \sum_{k=1}^{\infty} \frac{|y_k|^s |y_k|^{s^*/s^*}}{\|y\|_s^s/s^*} = \sum_{k=1}^{\infty} \frac{|y_k|^s}{\|y\|_s^s/s^*} = \|y\|_s. \tag{1.2.19}
$$

From (1.2.18) and (1.2.19) we deduce (1.2.17).

By (1.2.16) and (1.2.17), we obtain

$$
\|y\|_s = \left[ \sum_{n=1}^{N} u_n \left( \sum_{k=1}^{n} v_k w_k \right)^{r/s} \right]^{1/s} \leq K(r/s, 1) \sum_{n=1}^{N} v_k w_k^{r/s},
$$

which is equivalent to (1.2.6). This complete the Lemma 1.2.2.

**Proof of Lemma 1.2.3.** If $v = 0$, then our problem is trivial. We assume that $v \neq 0$. We need only to show that for $v_1 > 0$. Indeed, if $v_1 = 0$, the inequality (1.2.7) becomes

$$
\sum_{n=2}^{m} u_n \left( \sum_{k=2}^{n} v_k \right)^r \leq \sum_{k=2}^{m} v_k \text{ for } m = 2, \ldots, N.
$$
And (1.2.8) becomes
\[ \sum_{n=2}^{N} u_n \left( \sum_{k=2}^{n} v_k w_k \right)^r \leq \left( \frac{r}{r-1} \right) \sum_{k=2}^{N} v_k w_k^r. \]

They are equivalent \((N - 1)\)-tuples.

For \(v_1 > 0\), let
\[ y_n = \frac{\sum_{k=1}^{n} v_k w_k}{\sum_{k=1}^{n} v_k}. \]

We consider
\[ y_i - y_{i+1} = \frac{\sum_{k=1}^{i} v_k w_k}{\sum_{k=1}^{i} v_k} - \frac{\sum_{k=1}^{i+1} v_k w_k}{\sum_{k=1}^{i+1} v_k}. \]
\[ = \frac{\sum_{k=1}^{i} v_k w_k - v_{i+1} w_{i+1} \sum_{k=1}^{i} v_k}{\sum_{k=1}^{i} v_k \sum_{k=1}^{i+1} v_k}. \]
\[ = \frac{\sum_{k=1}^{i} v_{i+1} v_k (w_k - w_{i+1})}{\sum_{k=1}^{i} v_k \sum_{k=1}^{i+1} v_k} \geq 0. \]

Thus, \(y\) is decreasing. Let
\[ x_n = v_n - u_n \left( \sum_{k=1}^{n} v_k \right)^r. \]

We have
\[ \sum_{n=1}^{m} x_n = \sum_{n=1}^{m} v_n - \sum_{n=1}^{m} u_n \left( \sum_{k=1}^{n} v_k \right)^r \geq 0. \]

Let \(S_0 = 0\) and \(S_m = \sum_{n=1}^{m} x_n \geq 0\) for \(m = 1, ..., N\). We consider
\[ \sum_{n=1}^{N} x_n y_n^r = \sum_{n=1}^{N} (S_n - S_{n-1}) y_n^r \]
\[ = S_1 y_1^r + S_2 y_2^r - S_1 y_2^r + \cdots + S_N y_N^r - S_{N-1} y_{N-1}^r \]
\[ = S_1 (y_1^r - y_2^r) + S_2 (y_2^r - y_3^r) + \cdots + S_{N-1} (y_{N-1}^r - y_N^r) + S_N y_N^r. \]

Since \(y\) is decreasing and \(S_m \geq 0\) for \(m = 1, ..., N\) then
\[ \sum_{n=1}^{N} x_n y_n^r \geq 0. \]
That means
\[
\sum_{n=1}^{N} \left[ v_n - u_n \left( \sum_{k=1}^{n} v_k \right) \right] \left( \frac{\sum_{k=1}^{n} v_k w_k}{\sum_{k=1}^{n} v_k} \right)^r \geq 0,
\]
or
\[
\sum_{n=1}^{N} v_n \left( \frac{\sum_{k=1}^{n} v_k}{\sum_{k=1}^{n} v_k} \right)^r \geq \sum_{n=1}^{N} u_n \left( \sum_{k=1}^{n} v_k w_k \right)^r.
\]
Thus, we need only show that
\[
\sum_{n=1}^{N} v_n \left( \frac{\sum_{k=1}^{n} v_k}{\sum_{k=1}^{n} v_k} \right)^r \leq \left( \frac{r}{r-1} \right)^r \sum_{n=1}^{N} v_k w_k^r.
\]
It is the inequality of Theorem 1.2.1 Therefore, we complete Lemma 1.2.3.

From three above Lemma, we have shown that Theorem 1.2.2. We complete this section by showing that Theorem 1.2.2 fails when 0 < r ≤ s and also when 0 < s < 1, r > s.

In case: 0 < r ≤ s, we take \( v = (1,1,\ldots,1), w = (1,0,\ldots,0) \) and \( u_n = C n^{s-r-1} \).

The inequality (1.2.4) becomes
\[
C \sum_{n=1}^{m} n^{s-1} \leq m^s, \tag{1.2.20}
\]

inequality (1.2.5) becomes
\[
C \sum_{n=1}^{N} n^{s-1} \leq K(r,s). \tag{1.2.21}
\]

For \( s \geq 1 \), we have
\[
\sum_{n=1}^{m} n^{s-1} \leq m.m^{s-1} = m^s.
\]
We can choose \( C = 1 \) to (1.2.20) is satisfied.

For \( 0 < s < 1 \), we have
\[
\sum_{n=1}^{m} n^{s-1} \leq \int_{1}^{m+1} x^{s-1} dx = \frac{(m+1)^s - 1}{s}.
\]

We consider
\[
\frac{(m+1)^s - 1}{sm^s} = \frac{1}{s} \left[ \left( 1 + \frac{1}{m} \right)^s - \frac{1}{m^s} \right] \leq \frac{2^s}{s}.
\]

So that we can choose \( C = 2^{-s} s \) to (1.2.20) satisfied.

Since \( s > 0 \) then \( \sum_{n=1}^{\infty} n^{s-1} \) diverges. Hence, (1.2.21) fails for \( N \) large.
In second case: $0 < s < 1$ and $r > s$, we take $v_k = 2^k$, $w_k = v_k^{-s/r}$, $u_k = C v_k^{s-r}$. So that (1.2.4) becomes

$$C \sum_{n=1}^{m} 2^{n(s-r)} \left( \sum_{k=1}^{n} 2^k \right)^r \leq \left( \sum_{k=1}^{m} 2^k \right)^s = 2^s (2^m - 1)^s. \tag{1.2.22}$$

We have

$$\sum_{n=1}^{m} 2^{n(s-r)} \left( \sum_{k=1}^{n} 2^k \right)^r = \sum_{n=1}^{m} 2^{n(s-r)} 2^n (2^n - 1)^r \leq 2^r \sum_{n=1}^{m} 2^{n(s-r)} 2^{nr} = 2^r \sum_{n=1}^{m} 2^n s \leq \frac{2^{r+s}}{2^s - 1} 2^m (2^s - 1) \leq \frac{2^{r+s}}{2^s - 1} 2^s (2^m - 1)^s.$$

We can choose $C = (2^s - 1) 2^{-(r+s)}$ to (1.2.22) is satisfied so that (1.2.5) becomes

$$C \sum_{n=1}^{N} 2^{(s-r)n} \left( \sum_{k=1}^{n} 2^k 2^{-ks/r} \right)^r \leq K(r,s) \left( \sum_{k=1}^{N} 2^k 2^{-r} \right)^s = K(r,s) N^s. \tag{1.2.23}$$

We consider

$$\sum_{n=1}^{N} 2^{(s-r)n} \left( \sum_{k=1}^{n} 2^k 2^{-ks/r} \right)^r = \sum_{n=1}^{N} 2^{(s-r)n} \left( \frac{2^{(n+1)(1-s/r)} - 2^{1-s/r}}{2^{1-s/r} - 1} \right)^r = \left( \frac{2^{1-s/r}}{2^{1-s/r} - 1} \right)^r \sum_{n=1}^{N} 2^{(s-r)n} \left( 2^{(1-s/r)n} - 1 \right)^r.$$

We consider

$$\frac{[2^{(1-s/r)n} - 1]}{2^{(1-s/r)n}r} = \left( 1 - \frac{1}{2^{(1-s/r)n}} \right)^r \geq \left( 1 - \frac{1}{2^{1-s/r}} \right)^r.$$

So that

$$\sum_{n=1}^{N} 2^{(s-r)n} \left( \sum_{k=1}^{n} 2^k 2^{-ks/r} \right)^r \geq \left( \frac{2^{1-s/r}}{2^{1-s/r} - 1} \right)^r \left( 1 - \frac{1}{2^{1-s/r}} \right)^r \sum_{n=1}^{N} 2^{(s-r)n} 2^{nr-ns} = N.$$

Thus, the constant $C$ is chosen so that (1.2.22) is satisfied, as $N \to \infty$ the left-hand side of (1.2.23) grow like $N$ but the right-hand side like $N^s$ so that (1.2.23) fails for $N$ large.
Chapter 2

Factorable matrices

2.1 Factorable matrices

In this section we study the mapping properties of matrices $A = (a_{nk})_{n,k=1}^{\infty}$ of the type

$$a_{nk} = \begin{cases} a_n b_k & \text{when } 1 \leq k \leq n, \\ 0 & \text{otherwise}. \end{cases} \quad (2.1.1)$$

such the matrices have been called factorable. The matrices of $A$ may be complex numbers, but the norm of $A$ is unchanged when the entries are replaced by the absolute value. Thus we shall assume throughout this section that $a$'s and $b$'s are non-negative.

**Theorem 2.1.1.** Let $1 < p \leq q < \infty$, $a$ and $b$ be on non-negative numbers and let $A$ be the matrix given by (2.1.1). Then the following conditions are equivalent:

(i) $A$ is bounded from $l^p$ into $l^q$.

(ii) There exists $K_1$ such that for $m = 1, 2, \ldots$

$$\sum_{n=1}^{m} \left( a_n \sum_{k=1}^{n} b_k^p \right)^{q} \leq K_1 \left( \sum_{k=1}^{m} b_k^p \right)^{q/p}. \quad (2.1.2)$$

(iii) There exists $K_2$ such that for $m = 1, 2, \ldots$

$$\left( \sum_{n=m}^{\infty} a_n^q \right)^{1/q} \left( \sum_{k=1}^{m} b_k^p \right)^{1/p^*} \leq K_2. \quad (2.1.3)$$

(iv) There exists $K_3$ such that for $m = 1, 2, \ldots$

$$\sum_{k=m}^{\infty} \left( b_k \sum_{n=k}^{\infty} a_n^q \right)^{p^*} \leq K_3 \left( \sum_{n=m}^{\infty} a_n^q \right)^{p^*/q^*}. \quad (2.1.4)$$

**Proof.** Our proof follows the logical cycles

$$(ii) \Rightarrow (i) \Rightarrow (iii) \Rightarrow (ii) \quad \text{and} \quad (iv) \Rightarrow (i) \Rightarrow (iii) \Rightarrow (iv).$$

We need to prove only the first logical cycle, since both are equivalent. To see why is so, simply replace $(a_1, a_2, \ldots, a_N)$ by $(b_N, b_{N-1}, \ldots, b_1)$ and $(b_1, b_2, \ldots, b_N)$ by $(a_N, a_{N-1}, \ldots, a_1)$. 

17
The resulting factorable matrix, which we denote by \( A^s \), is the one obtained from \( A \) by reflection in the sinister diagonal. It follows, therefore, that

\[
\|A^s\|_{p^*,q^*} = \|A\|_{p,q}. \tag{2.1.5}
\]

If, further, we replace \( p \) by \( q^* \) and \( q \) by \( p^* \) we see that the hypothesis \( 1 < p \leq q < \infty \) and the condition \((i),(iii)\) are unchanged, while \((ii)\) and \((iv)\) are interchanged. Therefore, we need only to prove the equivalence of conditions \((i),(ii)\) and \((iii)\).

\((ii) \Rightarrow (i)\). Let \( x = (x_1,x_2,\ldots,x_N) \) with \( x_i \geq 0 \). If \((ii)\) holds, we apply the Theorem 1.2.2 with

\[
u_n = \frac{a_n^q}{K_1}, v_k = b_k^{p^*}, r = q, s = q/p.\]

Defining \( w \) by

\[
w_k = \begin{cases} x_k b_k^{1/(1-p)} & \text{if } b_k > 0, \\ 0 & \text{otherwise}. \end{cases}
\]

Inequality (1.2.5) gives

\[
\sum_{n=1}^N K_1 \left( \sum_{k=1}^n b_k^{p^*} x_k b_k^{1/(1-p)} \right)^q \leq K(q, q/p) \left[ \sum_{k=1}^N b_k^{p^*} \left( x_k b_k^{1/(1-p)} \right)^p \right]^{q/p},
\]

or

\[
\sum_{n=1}^N \left( \sum_{k=1}^n a_n b_k x_k \right)^q \leq K_1 K(q, q/p) \left( \sum_{k=1}^N x_k^p \right)^{q/p}.
\]

So that

\[
\|Ax\|_q^q \leq K_1 K(q, q/p) \|x\|_p^q = K_1(p^*)^q \|x\|_p.
\]

Thus, \((i)\) holds and

\[
\|A\|_{p,q} \leq p^* K_1^{1/q}. \tag{2.1.6}
\]

\((i) \Rightarrow (iii)\). Let \( m \) be a fixed \( 1 \leq m \leq N \) and suppose that \((i)\) holds. If \( x,y \) are \( N \)-tuples with non-negative. We have by Holder’s inequality

\[
\left| \sum_{n=1}^N a_n y_n \sum_{k=1}^n b_k x_k \right| \leq \left[ \sum_{n=1}^N \left( a_n \sum_{k=1}^n b_k x_k \right)^{q/q^*} \right]^{1/q^*} \left[ \sum_{n=1}^N y_n^{q^*} \right]^{1/q^*}
\]

\[
\leq \|Ax\|_q \|y\|_{q^*}
\]

\[
\leq \|A\|_{p,q} \|x\|_p \|y\|_{q^*}. \tag{2.1.7}
\]

Setting

\[
x_k = \begin{cases} b_k^{1/(p-1)} & \text{when } 1 \leq k \leq m, \\ 0 & \text{otherwise}, \end{cases}
\]

\[
y_n = \begin{cases} a_n^{q-1} & \text{when } m \leq n \leq N, \\ 0 & \text{otherwise}. \end{cases}
\]

So that (2.1.7) gives

\[
\sum_{n=m}^N a_n^q \sum_{k=1}^m b_k^{p^*} \leq \|A\|_{p,q} \left( \sum_{k=1}^m b_k^{p^*} \right)^{1/p} \left( \sum_{n=m}^N a_n^q \right)^{1/q^*}.
\]
Then
\[
\left( \sum_{n=m}^{N} a_n^q \right)^{1/q} \left( \sum_{k=1}^{m} b_k^p \right)^{1/p^*} \leq \|A\|_{p,q},
\]
so that (iii) holds with \( K_2 \leq \|A\|_{p,q} \).

(iii) ⇒ (ii). Let \( B_0 = 0 \) and \( B_n = \sum_{k=1}^{n} b_k^p \) for \( n = 1, 2, \ldots, N \). Applying the Power Rule, for \( q \geq 1 \) we get
\[
B_n^q = \left( \sum_{k=1}^{n} b_k^p \right)^q \leq q \sum_{k=1}^{n} b_k^p \left( \sum_{m=1}^{k} b_m^p \right)^{q-1} = q \sum_{k=1}^{n} b_k^p B_k^{q-1}.
\]
Then
\[
\sum_{n=1}^{m} q_n^q B_n^q \leq q \sum_{n=1}^{m} a_n^q \sum_{k=1}^{n} b_k^p B_k^{q-1} \leq q \sum_{k=1}^{m} b_k^p B_k^{q-1} \sum_{n=k}^{m} a_n^q \leq q \sum_{k=1}^{m} b_k^p B_k^{q-1} \sum_{n=k}^{N} a_n^q.
\]
Since (iii) holds, then
\[
\sum_{n=k}^{N} a_n^q \leq K_2^q \left( \sum_{k=1}^{m} b_k^p \right)^{-q/p^*} = K_2^q B_k^{-q/p^*}.
\]
Therefore,
\[
\sum_{n=1}^{m} a_n^q B_n^q \leq q K_2^q \sum_{k=1}^{m} b_k^p B_k^{q-1} B_k^{-q/p^*} \leq q K_2^q B_k^{q/p^*} \leq q K_2^q B_n^{p^*}.
\]
So that (ii) holds with \( K_1 \leq q K_2^q \). We complete Theorem 2.1.1.

2.2 Some Corollaries of Theorem 2.1.1

Corollary 2.2.1. Let \( p \) be the fixed such that \( 1 < p < \infty \). If
\[
a_n \sum_{k=1}^{n} b_k^p \leq K b_n^{1/(p-1)} \tag{2.2.1}
\]
for \( n = 1, 2, \ldots \), then \( A \) is bounded on \( l^p \) and
\[
\|A\|_{p,p} \leq K p^* \tag{2.2.2}
\]
The constant \( p^* \) is the best possible.
Proof. From inequality (2.2.1), we have
\[ \sum_{n=1}^{m} \left( a_n \sum_{k=1}^{n} b_k^{*} \right)^p \leq K \sum_{n=1}^{m} b_n^{p*}. \]

Applying Theorem 2.1.1 (ii) \( \Rightarrow \) (i) with \( p = q \), we obtain \( A \) is bounded on \( l^p \) and
\[ \|A\|_{p,p} \leq K p^*. \]

To prove that the \( p^* \) is best constant we will find \( a_n, b_k \) is satisfied (2.2.1) but
\[ \|A\|_{p,p} > C, \]
where \( 0 < C < K p^* \).

Let \( a_n = 1/n, b_k = 1 \). We have (2.2.1) always holds with \( K = 1 \). Thus, \( A \) is bounded \( l^p \) that means
\[ \|Ax\|_p = \sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{k=1}^{n} x_k \right)^p \leq p^* \sum_{n=1}^{\infty} x_n^p. \]

This is Hardy’s inequality. We will prove that the best constant in this part after we prove the Corollary 2.2.3.

By using the implication (iv) \( \Rightarrow \) (i) of Theorem 2.1.1 as similarly argument, we also give the following Corollary.

**Corollary 2.2.2.** Let \( p \) be the fixed and \( 1 < p < \infty \). If
\[ b_k \sum_{n=k}^{\infty} a_n^p \leq KA_k^{p-1} \]
for \( n = 1, 2, \ldots \), then \( A \) is bounded on \( l^p \) and
\[ \|A\|_{p,p} \leq K p. \]

The constant \( p \) is the best possible.

Now, we begin some inequalities of Hardy, Copson and Leinder. For these results we suppose that
\[ x_n \geq 0, \lambda > 0, \Lambda_n = \lambda_1 + \cdots + \lambda_n. \]

**Corollary 2.2.3.** If \( 1 < c \leq p \), then
\[ \sum_{n=1}^{\infty} \lambda_n \Lambda_n^{-c} \left( \sum_{k=1}^{n} \lambda_k x_k \right)^p \leq \left( \frac{p}{c-1} \right)^p \sum_{n=1}^{\infty} \lambda_n \Lambda_n^{p-c} x_n^p. \]

The constant is best possible.

**Proof.** Using the substitution
\[ a_n = \left( \frac{\lambda_n}{\Lambda_n^c} \right)^{1/p}, b_k = \frac{\lambda_k^{1/p^*}}{\Lambda_n^{1-c/p}}, y_n = \lambda_n \Lambda_n^{p-c} x_n^p, \]
The inequality (2.2.5) becomes

$$\sum_{n=1}^{\infty} a_n^p \left( \sum_{k=1}^{n} b_k y_k \right)^p \leq \left( \frac{p}{c-1} \right)^p \sum_{n=1}^{\infty} y_n^p.$$  

By hypothesis $1 < c \leq p$ so

$$0 < \frac{c-1}{p-1} < 1.$$  

Applying the Power Rule Lemma, we have

$$\sum_{k=1}^{n} b_k^p = \sum_{k=1}^{n} \lambda_k \Lambda_k^{(c-p)/(p-1)} = \sum_{k=1}^{n} \lambda_k \left( \sum_{j=1}^{k} \lambda_k \right)^{(c-1)/(p-1)-1} \leq \frac{p-1}{c-1} \Lambda_n^{(c-1)/(p-1)}.$$  

We also have

$$a_n^{-1} b_n^{1/(p-1)} = \left( \frac{\lambda_n}{\Lambda_n} \right)^{-1/p} \left( \frac{\lambda_n^{1/p}}{\Lambda_n^{1-c/p}} \right)^{1/(p-1)} = \frac{\lambda_n^{-1/p} \lambda_n^{1/p}}{\Lambda_n^{-c/p} \Lambda_n^{(1-c/p)/(p-1)}} = \Lambda_n^{(c-1)/(p-1)}.$$  

Therefore,

$$\sum_{k=1}^{n} b_k^p \leq \frac{p-1}{c-1} a_n^{-1} b_n^{1/(p-1)}.$$  

Thus, Corollary (2.2.1) may be apply with $K = (p-1)/(c-1)$ to give $A$ is bounded on $l^p$ and

$$\|A\|_{p,p} \leq K p^* = \frac{p}{c-1}.$$  

Now, we’ll prove that the constant is best possible. We take

$$x_n = \begin{cases} n^{(c-p-1)/p} & \text{when } 1 \leq n \leq N, \\ 0 & \text{otherwise.} \end{cases}$$  

And

$$\lambda_n = \begin{cases} 1 & \text{when } 1 \leq n \leq N, \\ 0 & \text{otherwise.} \end{cases}$$  

Inequality (2.2.5) becomes

$$\sum_{n=1}^{N} n^{-c} \left( \sum_{k=1}^{n} n^{(c-p-1)/p} \right)^p \leq \left( \frac{p}{c-1} \right)^p \sum_{n=1}^{N} \frac{1}{n}. \quad (2.2.6)$$
And we have
\[\sum_{k=1}^{n} k^{(c-p-1)/p} > \int_1^{n} x^{(c-p-1)/p} dx = \frac{p}{c-1} \left(n^{(c-1)/p} - 1\right).\]

Therefore,
\[n^{-c} \left(\sum_{k=1}^{n} n^{(c-p-1)/p}\right)^p > \left(\frac{p}{c-1}\right)^p n^{-c-1} \left(1 - \frac{1}{n^{(c-1)/p}}\right)^p = \left(\frac{p}{c-1}\right)^p n^{-1} \left(1 - \frac{p}{n^{(c-1)/p}}\right).\]

Summing on \(n\), we obtain
\[\sum_{n=1}^{N} n^{-c} \left(\sum_{k=1}^{n} n^{(c-p-1)/p}\right)^p > \left(\frac{p}{c-1}\right)^p \sum_{n=1}^{N} \left(1 - \frac{p}{n^{(c-1)/p}}\right) \frac{1}{n}.\]

Then
\[\sum_{n=1}^{N} n^{-c} \left(\sum_{k=1}^{n} n^{(c-p-1)/p}\right)^p > \left(\frac{p}{c-1}\right)^p \left[1 - \frac{p}{\sum_{n=1}^{N} n^{(1-c-p)/p}}\right].\]

Since \(1 < c \leq p\) so \((c + p - 1)/p > 1\). Thus,
\[\sum_{n=1}^{N} n^{(1-c-p)/p}\] converges as \(N \to \infty\).

In addition, we have \(\sum_{n=1}^{N} n^{-1} \to \infty\) as \(N \to \infty\). Hence, the right hand side of the above inequality tend to \(\left(\frac{p}{c-1}\right)^p\) as \(N\) tend to infinite . We complete Corollary 2.2.3.

Now, we come back the Corollary 2.2.1 to show that the best constant. From the Corollary 2.2.3 we find the \(K = (p - 1)/(c - 1)\). We give
\[c = \frac{p - 1}{K} + 1.\]

Thus, we can choose \(a_n, b_k\)
\[a_n = \begin{cases} n^{(1-K-p)/(Kp)} & \text{when } 1 \leq n \leq N, \\ 0 & \text{otherwise}, \end{cases}\]
\[b_n = \begin{cases} n^{(K+p-1)/(Kp)-1} & \text{when } 1 \leq n \leq N, \\ 0 & \text{otherwise}. \end{cases}\]

And
\[x_n = \begin{cases} n^{-1/p} & \text{when } 1 \leq n \leq N, \\ 0 & \text{otherwise}, \end{cases}\]
where N is choose later. We have \( a_n \) and \( b_k \) are satisfied the hypothesis of Corollary 2.2.1. Indeed,

\[
\sum_{k=1}^{n} a_n b_k^{*} = \sum_{k=1}^{n} \left[ (Kp-1)/(Kp) \right] a_n b_k^{*} < \sum_{k=1}^{n} \left[ (Kp-1)/(Kp) \right] b_k^{*} = \sum_{k=1}^{n} \left[ (Kp-1)/(Kp) \right] b_k^{*} \]

We have

\[
\sum_{k=1}^{n} b_k x_k = \sum_{k=1}^{n} \left[ \frac{Kp}{p-1} \left( n^{(p-1)/(Kp)} - 1 \right) \right]
\]

Thus,

\[
\|Ax\|_p = \sum_{n=1}^{N} a_n^p \left( \sum_{k=1}^{n} b_k x_k \right)^p
\]

\[
= \sum_{n=1}^{N} \left[ \frac{Kp}{p-1} \left( n^{(p-1)/(Kp)} - 1 \right) \right]^p
\]

\[
= (Kp^*)^p \sum_{n=1}^{N} \left[ \frac{p}{n^{(p-1)/(Kp)}} \right] \left[ \sum_{n=1}^{N} \left( n^{-1/(Kp^*)} \right) \right] \left[ \frac{1}{K} \sum_{k=1}^{n} \right] \frac{1}{n}
\]

Thus,

\[
\|A\|_p > Kp^* I^{1/p}.
\]
CHAPTER 2. FACTORABLE MATRICES

We have \( \sum_{n=1}^{N} n^{-1/(Kp^*)+1} \) converges as \( N \to \infty \) and \( \sum_{k=1}^{N} n^{-1} \) is divergent as \( N \to \infty \). Therefore,

\[
I = \left[ 1 - \frac{p \sum_{n=1}^{N} n^{-1/(Kp^*)+1}}{\sum_{k=1}^{N} \frac{1}{n}} \right] \to 1 \quad \text{as} \quad N \to \infty.
\]

Thus, \( 0 < C = Kp^* I^{1/p} < Kp^* \). We have shown that the best possible constant in Corollary 2.2.1.

**Corollary 2.2.4.** If \( 0 \leq c < 1 \), then

\[
\sum_{n=1}^{\infty} \lambda_n \Lambda_n^{-c} \left( \sum_{k=n}^{\infty} \lambda_k x_k \right)^p \leq \left( \frac{p}{1-c} \right)^p \sum_{n=1}^{\infty} \lambda_n \Lambda_n^{p-c} x_n^p. \quad (2.2.7)
\]

The constant is best possible.

**Proof.** We also substitute \( y_n^p = \lambda_n \Lambda_n^{p-c} x_n^p \) and take \( A' \) give by

\[
a_n = \frac{\lambda_n^{1/p^*}}{\Lambda_n^{1-c/p}}, \quad b_k = \left( \frac{\lambda_k}{\Lambda_k^c} \right)^{1/p}.
\]

By Power Rule inequality, we have

\[
\sum_{k=1}^{n} b_k^p = \sum_{k=1}^{n} \frac{\lambda_k \Lambda_k^{-c}}{1-c} \leq \frac{1}{1-c} \Lambda_n^{1-c}.
\]

In addition,

\[
a_n^{-1} b_n^{1/(p^*-1)} = \frac{\lambda_n^{-1/p^*}}{\Lambda_n^{-1+c/p}} \left( \frac{\lambda_n}{\Lambda_n^c} \right)^{(p-1)/p} \]
\[
= \frac{\lambda_n^{-1/p^*}}{\Lambda_n^{-1+c/p}} \Lambda_n^{c(p-1)/p}
\]
\[
= \Lambda_n^{1-c}.
\]

Hence,

\[
\sum_{k=1}^{n} b_k^p \leq \frac{1}{1-c} a_n^{-1} b_n^{1/(p^*-1)}.
\]

By Corollary 2.2.1 we get \( A' \) is bounded on \( p^{p^*} \) and

\[
\|A'\|_{p^*,p^*} \leq \frac{p}{1-c}.
\]

Therefore,

\[
\|A\|_{p,p} \leq \frac{p}{1-c}.
\]

We complete Corollary 2.2.4. \( \square \)
After that, we also have some similarly inequalities with \( \Lambda_n^* = \sum_{k=n}^\infty \lambda_n \) for \( n = 1, 2, \ldots \)

**Corollary 2.2.5.** If \( 0 \leq c < 1 \), then
\[
\sum_{n=1}^\infty \lambda_n (\Lambda_n^*)^{-c} \left( \sum_{k=n}^\infty \lambda_k x_k \right)^p \leq \left( \frac{p}{1-c} \right)^p \sum_{n=1}^\infty \lambda_n (\Lambda_n^*)^{-c} x_n^p. \tag{2.2.8}
\]
The constant is best possible.

**Corollary 2.2.6.** If \( 1 < c \leq p \),
\[
\sum_{n=1}^\infty \lambda_n (\Lambda_n^*)^{-c} \left( \sum_{k=n}^\infty \lambda_k x_k \right)^p \leq \left( \frac{p}{c-1} \right)^p \sum_{n=1}^\infty \lambda_n (\Lambda_n^*)^{-c} x_n^p. \tag{2.2.9}
\]
The constant is best possible.

**Corollary 2.2.7.** If \( 1 < p \leq q < \infty \) and \( c > 1 \), then
\[
\sum_{n=1}^\infty \lambda_n \Lambda_n^{(1-c)q/p-1} \left( \sum_{k=1}^n \lambda_k x_k \right)^q \leq K \left( \sum_{n=1}^\infty \lambda_n \Lambda_n^{p-c} x_n^p \right)^{q/p}. \tag{2.2.10}
\]

**Proof.** Let
\[
a_n^q = \lambda_n \Lambda_n^{(1-c)q/p-1}, \quad b_k = \lambda_k^{1/p} \Lambda_k^{c/p-1}, \quad y_n^p = \lambda_n \Lambda_n^{p-c} x_n^p.
\]
Inequality (2.2.10) becomes
\[
\sum_{n=1}^\infty \left( a_n \sum_{k=1}^n b_k y_k \right)^q \leq K \left( \sum_{n=1}^\infty y_n^p \right)^{q/p}.
\]
Applying Theorem 2.1.1 \((ii) \Rightarrow (i)\), we need only to point out
\[
\sum_{n=1}^m \left( a_n \sum_{k=1}^n b_k^p \right)^q \leq K_1 \left( \sum_{n=1}^m b_k^p \right)^{q/p}.
\]
We consider
\[
\sum_{k=1}^n b_k^p = \sum_{k=1}^n \lambda_k \Lambda_k^{(c-p)/(p-1)}
\leq K \left( \sum_{k=1}^n \lambda_k \right)^{(c-1)/(p-1)} \quad \text{(By Power Rule Lemma )}.
\]
Thus,
\[ \sum_{n=1}^{m} \left( a_n \sum_{k=1}^{n} b_k^{p^*} \right)^q \leq K^q \sum_{n=1}^{m} a_n^q \Lambda_n^{(c-1)q/(p-1)} \]
\[ = K^q \sum_{n=1}^{m} \lambda_n \Lambda_n^{(1-c)q/p-1} \Lambda_n^{(c-1)q/(p-1)} \]
\[ = K^q \sum_{n=1}^{m} \lambda_n \Lambda_n^{(c-1)q/p - 1} \] (By Power Rule Lemma)
\[ \leq C_1 \left( \Lambda_m^{(c-1)/(p-1)} \right)^{q/p} \]
\[ \leq C_2 \left( \sum_{n=1}^{m} b_k^{p^*} \right)^{q/p} \]
where \( C_1, C_2 \) are constants. We complete Corollary 2.2.7.

2.3 Example

Our next result gives a complete description of the mapping properties of factorable matrices of the form
\[ a_n = n^{-\alpha}, \quad b_k = k^{-\beta}. \] (2.3.1)

**Corollary 2.3.1.** Let \( 1 \leq p, q \leq \infty \) and let \( A \) be the matrix given by (2.3.1). Then \( A \) is bounded from \( l^p \) into \( l^q \) if and only if

(i) \( \alpha \geq 0 \) and \( \alpha + \beta \geq 0 \) \((p = 1, q = \infty)\),

(ii) \( \alpha > 0 \) and \( \alpha + \beta \geq \frac{1}{p^*} \) or \( \alpha = 0 \) and \( \beta > \frac{1}{p^*} \) \((p > 1, q = \infty)\),

(iii) \( \alpha > \frac{1}{q} \) and \( \alpha + \beta \geq \frac{1}{q} + \frac{1}{p^*} \) \((1 < p \leq q < \infty)\),

(iv) \( \alpha > \frac{1}{q} \) and \( \alpha + \beta > \frac{1}{q} + \frac{1}{p^*} \) \((p > q)\).

Moreover, \( \| A \|_{1,\infty} = 1 \). And in the case \( 1 < p < \infty, q = \infty, \alpha = 0, \beta > 1/p^* \) we have
\[ \| A \|_{p,\infty} = \| k^{-\beta} \|_{p^*}. \]

Before proving, we have a notation is used in proof of the above Corollary. We estimate the following sum
\[ \sum_{k=1}^{n} n^{-\alpha} \sim \begin{cases} n^{1-\alpha} & \text{if } 0 < \alpha < 1, \\ \ln n & \text{if } \alpha = 1. \end{cases} \]

**Proof.** We also have
\[ (Ax)_m = y_m = m^{-\alpha} \sum_{k=1}^{m} k^{-\beta} x_k. \]
First, we will show that $A$ is bounded from $l^1$ to $l^\infty \iff (i)$.  

($\Rightarrow$) Let $x = (1, 0, \ldots)$ so that $y_m = m^{-\alpha}$. Since $A$ is bounded from $l^1$ in $l^\infty$ so that $Ax \in l^p$. Thus, 

$$
\sup_{m \in \mathbb{N}} m^{-\alpha} < +\infty.
$$

That means $\alpha \geq 0$. Now, we take $x^{(n)} = (0, \ldots, 0, 1, 0, \ldots)$ so 

$$
(y_m^{(n)}) = \begin{cases} 
m^{-\alpha} n^{-\beta} & \text{for } m > n, \\
n^{-\alpha + \beta} & \text{for } m = n, \\
0 & \text{for } m < n.
\end{cases}
$$

Since $\alpha \geq 0$, we get 

$$
\|Ax^{(n)}\|_\infty = n^{-(\alpha + \beta)}.
$$

Because $A$ is bounded, we have for all $n$ 

$$
n^{-(\alpha + \beta)} = \|Ax^{(n)}\|_\infty \leq \|A\|_{1,\infty} \|x^{(n)}\|_1 = \|A\|_{1,\infty}.
$$

Then $\alpha + \beta \geq 0$.  

($\Leftarrow$) For every $m \in \mathbb{N},$ 

$$
\left| m^{-\alpha} \sum_{k=1}^{m} k^{-\beta} x_k \right| \leq \sum_{k=1}^{m} k^{-(\alpha + \beta)}|x_k| \quad (\text{Since } \alpha \geq 0)
$$

$$
\leq \sum_{k=1}^{m} |x_k| \quad (\text{Since } \alpha + \beta \geq 0)
$$

$$
\leq \|x\|_1.
$$

Hence, $\|Ax\|_\infty \leq \|x\|_1$ and  

$$
\|A\|_{1,\infty} \leq 1.
$$

Then $\|Ax\|_\infty \leq \|A\|_{1,\infty} \|x_0\|_1.$

We claim that 

$$
\|A\|_{1,\infty} = 1.
$$

Indeed, we choose $x_0 = (1, 0, \ldots)$ so that 

$$
\|Ax_0\|_\infty = \sup_{m \in \mathbb{N}} m^{-\alpha} = 1
$$

and $\|x_0\|_1 = 1$. Since 

$$
\|Ax_0\|_\infty \leq \|A\|_{1,\infty} \|x_0\|_1.
$$

Then $\|A\|_{1,\infty} \geq 1$ with choosing above $x_0$.  

Combining (2.3.2), we complete the claim.

(b) Next, we will show that $A$ is bounded from $l^p$ to $l^\infty \iff (ii)$ with $1 < p < \infty$. 

($\Rightarrow$) Similarly part (a), we also let $x = (1, 0 \ldots)$ and we give $\alpha \geq 0$.  

For $\alpha \geq 0$, we choose $x^{(n)} = (1, 2, \ldots, n, 0 \ldots)$ then 

$$
(Ax^{(n)})_m = \begin{cases} 
m^{-\alpha} \sum_{k=1}^{n} k^{-\beta + 1} \sim m^{-\alpha} n^{2-\beta} & \text{for } m \geq n, \\
m^{-\alpha} \sum_{k=1}^{m} k^{-\beta + 1} \sim m^{-\alpha} m^{2-\beta} & \text{for } m < n.
\end{cases}
$$

Hence, $\|Ax^{(n)}\|_\infty = m^{2-\beta}$.  

Because $A$ is bounded, we have for all $n$ 

$$
n^{2-\beta} = \|Ax^{(n)}\|_\infty \leq \|A\|_{1,\infty} \|x^{(n)}\|_1 = \|A\|_{1,\infty}.$$

Then $\|A\|_{1,\infty} \geq 1$ with choosing above $x_0$.  

Combining (2.3.2), we complete the claim.
We have
\[ \sup_{m \geq n} m^{-\alpha} n^{2-\beta} = n^{2-\alpha-\beta} \leq \|Ax\|_\infty. \quad (2.3.3) \]

We also have \( A \) is bounded so
\[ \|Ax(n)\|_\infty \leq \|A\|_{p,\infty} \|x(n)\|_p = \|A\|_{p,\infty} \left( \sum_{k=1}^{n} k^p \right)^{1/p} \leq \|A\|_{p,\infty} n^{(p+1)/p}. \quad (2.3.4) \]

From (2.3.3) and (2.3.4), we obtain \( 2 - \alpha - \beta \leq (p + 1)/p \) that means \( \alpha + \beta \geq 1/p^* \).

For \( \alpha = 0 \), we have \( \beta \geq 1/p^* \) and
\[ (Ax(n))_m = \sum_{k=1}^{m} k^{-\beta} x_k. \]

We need only to point out that if \( \beta = 1/p^* \), \( A \) is not bounded from \( l^p \) into \( l^\infty \).

Let \( x(n) = (1^{-1/p}, \ldots, n^{-1/p}, 0, \ldots) \) so
\[ (Ax(n))_m = \begin{cases} \sum_{k=1}^{n} k^{-1} \sim \ln n & \text{for } m \geq n, \\ \sum_{k=1}^{m} k^{-1} = \ln m & \text{for } m < n. \end{cases} \]

Thus, \( \|Ax(n)\|_\infty \sim \ln n \) and \( \|x(n)\|_p = \left( \sum_{k=1}^{n} k^{-1} \right)^{1/p} \sim (\ln n)^{1/p} \).

Since for \( 1 < p < \infty \)
\[ \lim_{n \to \infty} \frac{\ln n}{(\ln n)^{1/p}} = \infty. \]

Therefore, \( A \) is not bounded from \( l^p \) in \( l^\infty \).

(\( \Leftarrow \)) We will prove that if \( \alpha > 0 \) and \( \alpha + \beta \geq 1/p^* \), \( A \) is bounded from \( l^p \) in \( l^\infty \) with \( 1 < p < \infty \).

We consider
\[ |(Ax(n))_m| = \left| m^{-\alpha} \sum_{k=1}^{m} k^{-\beta} x_k \right| \leq \sum_{k=1}^{m} k^{-(\alpha+\beta)} |x_k| \]
\[ \leq \left( \sum_{k=1}^{m} k^{-(\alpha+\beta)p^*} \right)^{1/p^*} \left( \sum_{k=1}^{m} |x_k|^p \right)^{1/p} \]
\[ \leq K \|x\|_p. \]

Then \( A \) is bounded.
We continue to prove that if $\alpha = 0$ and $\beta > 1/p^*$, $A$ is bounded $l^p$ to $l^\infty$ with $p > 1$.

\[
(Ax^{(n)})_m = \left| \sum_{k=1}^{m} k^{-\beta} x_k \right|
\]

\[
\leq \sum_{k=1}^{m} k^{-\beta} |x_k|
\]

\[
\leq \left( \sum_{k=1}^{m} k^{-\beta p^*} \right)^{1/p^*} \left( \sum_{k=1}^{m} x_k^p \right)^{1/p}
\]

\[
\leq \|(k^{-\beta})\|_{p^*} \|x\|_p.
\]

Then $A$ is bounded and $\|A\|_{p,\infty} \leq \|(k^{-\beta})\|_{p^*}$. In this case, we can show that

\[
\|A\|_{p,\infty} = \|k^{-\beta}\|_{p^*}.
\]

By taking

\[
x^{(n)} = (1^{-\beta(p^*-1)}, \ldots, n^{-\beta(p^*-1)}, 0, \ldots).
\]

We have $(Ax^{(n)})_m = \sum_{k=1}^{m} k^{-\beta p^*}$. Hence, we obtain

\[
\|Ax^{(n)}\|_\infty = \sum_{k=1}^{m} k^{-\beta p^*}
\]

\[
= \left( \sum_{k=1}^{m} k^{-\beta p^*} \right)^{1/p^*} \left( \sum_{k=1}^{m} k^{-\beta p^*} \right)^{1/p}
\]

\[
= \left( \sum_{k=1}^{m} k^{-\beta p^*} \right)^{1/p^*} \left( \sum_{k=1}^{m} k^{-\beta(p^*-1)p} \right)^{1/p}
\]

\[
= \|(k^{-\beta})\|_{p^*} \|x^{(n)}\|_p.
\]

In addition, $\|A\|_{p,\infty} \leq \|(k^{-\beta})\|_{p^*}$. We get (2.3.5).

(c) After that, we will prove the equivalence of $(iii)$ and $A$ is bounded from $l^p$ into $l^q$ with $1 < p \leq q < \infty$.

(⇒) We choose $x = (1, 0, \ldots)$, then

\[
\|Ax\|_q = \sum_{m=1}^{\infty} m^{-\alpha q} \left( \sum_{k=1}^{m} k^{-\beta} x_k \right)^q = \sum_{m=1}^{\infty} m^{-\alpha q}.
\]

Since $Ax \in l^q$ then

\[
\sum_{m=1}^{\infty} m^{-\alpha q} < +\infty.
\]

Hence, $\alpha q > 1$. We assume that

\[
\alpha + \beta < \frac{1}{q} + \frac{1}{p^*}.
\]

(2.3.6)
Since \( \alpha > 1/q \), we deduce
\[
\beta + \frac{1}{p} < 1.
\]

Let \( x^{(N)} = (N^{-1/p}, \ldots, N^{-1/p}, 0, \ldots) \). We get
\[
(Ax^{(N)})_m = \begin{cases} 
m^{-\alpha} \sum_{k=1}^{m} k^{-(1/p+\beta)} & \text{for } m < N, \\
m^{-\alpha} \sum_{k=1}^{N} k^{-(1/p+\beta)} & \text{for } m \geq N.
\end{cases}
\]

Then,
\[
\|Ax^{(N)}\|_q^q = \sum_{m=1}^{N} m^{-\alpha q} \left( \sum_{k=1}^{m} k^{-(1/p+\beta)} \right)^q + \sum_{m=N+1}^{\infty} m^{-\alpha q} \left( \sum_{k=1}^{N} k^{-(1/p+\beta)} \right)^q
\]
\[
\geq \sum_{m=1}^{N} m^{-\alpha q} \left( \sum_{k=1}^{m} k^{-(1/p+\beta)} \right)^q
\]
\[
\sim \sum_{m=1}^{N} m^{-\alpha q} m^{(1-1/p-\beta)q} \quad \text{(Since } 0 < \beta + \frac{1}{p} < 1).\]

Since
\[
\alpha q + (\beta + \frac{1}{p} - 1)q < 1.
\]

Then
\[
\|Ax^{(N)}\|_q^q \sim N^{1-\alpha q_m} m^{(1-1/p-\beta)q}.
\]

On other hand,
\[
\|x^{(N)}\|_p^q = \left( \sum_{m=1}^{N} m^{-1} \right)^{q/p} \sim (\ln N)^{q/p}.
\]

Thus,
\[
\lim_{N \to \infty} \frac{\|Ax^{(N)}\|_q^q}{\|x^{(N)}\|_p^q} = +\infty.
\]

Hence, \( A \) is not bounded that this is opposite the hypothesis. Hence we get the proof.

(\( \Leftarrow \)) Applying Theorem 2.1.1, so we need only show that
\[
\sum_{n=1}^{m} n^{-\alpha q} \left( \sum_{k=1}^{n} k^{-\beta p^*} \right)^q \leq K \left( \sum_{n=1}^{m} n^{-\beta p^*} \right)^{q/p}.
\]
Since \( \alpha + \beta \geq \frac{1}{q} + \frac{1}{p} \) so \( \beta p^* \geq 1 + p^*(1/q - \alpha) \). Therefore,

\[
\sum_{n=1}^{m} n^{-\alpha q} \left( \sum_{k=1}^{n} n^{-\beta p^*} \right)^{q} \leq \sum_{n=1}^{m} n^{-\alpha q} \left( \sum_{k=1}^{n} k^p(\alpha - 1/q - 1) \right)^{q} \\
\sim \sum_{n=1}^{m} n^{-\alpha q} n^p(\alpha - 1/q) \quad \text{(Since } p^*(\alpha - 1/q) - 1 < -1 \text{)} \\
= \sum_{n=1}^{m} n^{-\alpha q + p^*(\alpha - 1)} \\
\sim m^{1-\alpha q + p^*(\alpha - 1)} \quad \text{(Since } \alpha - p^*(\alpha - 1) < 1 \text{)} \\
= m^{(\alpha q - 1)/(p - 1)}.
\]

On the other hand,

\[
\left( \sum_{k=1}^{m} n^{-\beta p^*} \right)^{q/p} \geq Km^p(\alpha - 1/q)q/p = Km^p(\alpha - 1)/p = Km^{(\alpha q - 1)/(p - 1)},
\]

where \( K \) is constant. Thus, \( A \) is bounded from \( l^p \) into \( l^q \).

\( (d) \) Finally, we will show that \( A \) is bounded form \( l^p \) into \( l^q \leftrightarrow (iv) \) with \( 1 < q < p < \infty \).

\( (\Rightarrow) \) Similarly part \( (c) \), we also take \( x = (1, 0, \ldots) \) and we obtain \( \alpha > 1/q \).

Next, we suppose that

\[
\alpha + \beta \leq \frac{1}{q} + \frac{1}{p^*}.
\]

We use inequality

\[
| < Ax, y > | \leq \| A \|_{p,q} \| x \|_p \| y \|_{q^*}.
\]

(2.3.7)

Let \( x^{(N)} = (1^{-1/p}, \ldots, N^{-1/p}, 0, \ldots) \) and \( y^{(N)} = (1^{-1/q^*}, \ldots, N^{-1/q^*}, 0, \ldots) \).

\[
\| x^{(N)} \|_p \| y^{(N)} \|_{q^*} = \left( \sum_{k=1}^{N} k^{-1} \right)^{1/p} \left( \sum_{k=1}^{N} k^{-1} \right)^{1/q^*} \sim (\ln N)^{1/p + 1/q^*}.
\]

And

\[
< Ax^{(N)}, y^{(N)} > = \sum_{m=1}^{N} m^{-(\alpha + 1/q^*)} \sum_{k=1}^{m} k^{-(\beta + 1/p)} \sim \sum_{m=1}^{N} m^{-(\alpha + 1/q^*)} m^{1-\beta - 1/p} = I_2.
\]

If \( \alpha + \beta = 1/q + 1/p^* \), we have

\[
1 - \alpha - \beta - \frac{1}{p} - \frac{1}{q^*} = -1.
\]

Thus

\[
| < Ax^{(N)}, y^{(N)} > | > I_2 \sim (\ln N).
\]

Since \( p > q \) so that

\[
0 \leq \frac{1}{p} + \frac{1}{q^*} = \frac{1}{p} + \frac{q - 1}{q} = 1 + \frac{1}{p} - \frac{1}{q} < 1.
\]
We have \( \ln N > (\ln N)^{1/p+1/q^*} \). Therefore,

\[
\lim_{N \to \infty} \frac{|<Ax(N), y(N)>|}{\|x(N)\|_p \|y(N)\|_{q^*}} = +\infty.
\]

Thus, (2.3.7) fails as \( N \) tends to infinity. That means \( A \) is not bounded. If \( \alpha + \beta < 1/q + 1/p^* \), we deduce

\[1 - \alpha - \beta - \frac{1}{p} - \frac{1}{q^*} > -1.\]

Then \( I_2 \sim N^{-\alpha-\beta-1/q-1/p^*} \). Thus,

\[
\lim_{N \to \infty} \frac{|<Ax(N), y(N)>|}{\|x(N)\|_p \|y(N)\|_{q^*}} = +\infty.
\]

Therefore, (2.3.7) is not true as \( N \to \infty \).

Summary, if \( \alpha > \frac{1}{q} \) and \( \alpha + \beta \leq \frac{1}{q} + \frac{1}{p^*} \), then we get \( A \) is not bounded which it is opposite the hypothesis. We complete this part.

(\( \Leftarrow \)) Let \( \gamma = 1 - \alpha \). Applying part (c) with \( p = q \) to the factorable matrix \( A' \) where

\[a'_n = n^{-\alpha}, b'_k = k^{-\gamma}.\]

We obtain \( A' \) is bounded on \( l^q \). On the other hand, the matrix \( A \) may be factored:

\[A = A'D,\]

where \( D \) is the diagonal matrix with diagonal entries:

\[d_k = k^{\gamma-\beta}.\]

Sine \( \alpha + \beta > 1/q + 1/p^* \), we get

\[
\frac{(\gamma - \beta) pq}{p - q} = \frac{(1 - \alpha - \beta) pq}{p - q} < (1 - \frac{1}{p} - \frac{1}{q^*})pq(p - q) = -1 \tag{2.3.8}
\]

then \( d \in \{pq/(p-q)\} \). It follows from Holder’s inequality, that \( D \) is bounded \( l^p \) into \( l^q \) and thus,

\[
\|Ax\|_q = \|A'Dx\|_q \leq \|A'\|_{q,q} \|Dx\|_q \leq \|A'\|_{q,q} \|D\|_{p,q} \|x\|_p.
\]

Hence \( A \) is bounded from \( l^p \) into \( l^q \). \( \square \)
3.1 Weighted mean matrices

In this section we obtain simple necessary and sufficient conditions for a weighted mean matrix to map $l^p$ into $l^q$ ($1 \leq p, q \leq \infty$). We recall that a weighted matrix, $A$ is an infinite matrix of the form

$$a_{nk} = \begin{cases} a_k/A_n & \text{when } 1 \leq k \leq n, \\ 0 & \text{otherwise}. \end{cases}$$

(3.1.1)

where $a_n$’s is non-negative numbers with partial sums $A_n = a_1 + a_2 + \cdots + a_n$. We claim that

$$\|A\|_{p,\infty} = 1 \quad (1 \leq p < \infty),$$

(3.1.2)

$$\|A\|_{1,q} = \sup_k a_k \left( \sum_{n=k}^{\infty} A_n^{-q} \right)^{1/q} \quad (1 \leq q < \infty).$$

(3.1.3)

First, we will prove that (3.1.2). We have

$$(Ax)_m = \begin{cases} A_n^{-1} \sum_{k=1}^{m} a_k x_k & \text{when } 1 \leq m < n, \\ A_n^{-1} \sum_{k=1}^{n} a_k x_k & \text{when } m = n, \\ 0 & \text{otherwise}. \end{cases}$$

Then,

$$\|Ax\|_{\infty} = A_n^{-1} \sum_{k=1}^{n} a_k x_k$$

$$\leq A_n^{-1} \left( \sum_{k=1}^{n} a_k^{p^*} \right)^{1/p^*} \left( \sum_{k=1}^{n} x_k^{p^*} \right)^{1/p} \quad \text{(By Holder’s inequality)}$$

$$\leq A_n^{-1} \left( \sum_{k=1}^{n} a_k^{p^*} \right)^{1/p^*} \|x\|_p.$$  

(3.1.4)

Since $p \geq 1$, then $p^* \geq 1$ and we have

$$\sum_{k=1}^{n} a_k^{p^*} \leq \left( \sum_{k=1}^{n} a_k \right)^{p^*} = A_n^{p^*}.$$
So that
\[ A_n^{-1} \left( \sum_{k=1}^{n} a_k^p \right)^{1/p'} \leq 1 \] (3.1.5)

From (3.1.4) and (3.1.5) we give
\[ \|Ax\|_\infty \leq \|x\|_p. \]
Thus,
\[ \|A\|_{p,\infty} \leq 1. \] (3.1.6)

Let \( x_0 = (1, 0, \ldots), \) we have \( Ax_0 = (1, A_2^{-1}a_1, \ldots, A_n^{-1}a_1, 0, \ldots)^T. \) Hence we obtain
\[ \|Ax_0\|_{p,\infty} = 1 = \|x_0\|_p. \]
Therefore, \( \|A\|_{p,\infty} = 1. \)

Next, we will prove that (3.1.3).
Let \( x^{(k)} = (0, \ldots, 0, 1, 0, \ldots), \) we have
\[ (Ax^{(k)})_m = \begin{cases} 0 & \text{for } 1 \leq m < k, \\
A_m^{-1}a_k & \text{for } k \leq m \leq n, \\
0 & \text{otherwise.} \end{cases} \]
Thus,
\[ \|Ax\|_q = \left[ \sum_{m=k}^{n} \left( A_m^{-1}a_k \right)^q \right]^{1/q} = a_k \left( \sum_{m=k}^{n} A_m^{-q} \right)^{1/q} \leq a_k \left( \sum_{m=k}^{\infty} A_m^{-q} \right)^{1/q} \leq \sup_{k \in \mathbb{N}} a_k \left( \sum_{m=k}^{\infty} A_m^{-q} \right)^{1/q}. \]

And we also have \( \|x^{(k)}\|_1 = 1, \) we give
\[ \|A\|_{1,q} \leq \sup_{k \in \mathbb{N}} a_k \left( \sum_{n=k}^{\infty} A_n^{-q} \right)^{1/q}. \] (3.1.7)

We take
\[ f^{(k)}_n = \begin{cases} A_n^{-1}a_kx_k & \text{when } 1 \leq k \leq n, \\
0 & \text{when } k > n. \end{cases} \]
We have

\[
\| f^{(k)} \|_q = \left( \sum_{n=1}^{\infty} |f_n^{(k)}|^q \right)^{1/q} = \left( \sum_{n=1}^{k-1} |f_n^{(k)}|^q + \sum_{n=k}^{\infty} |f_n^{(k)}|^q \right)^{1/q} = \left[ \sum_{n=k}^{\infty} \left( A_n^{-1} a_k |x_k| \right)^q \right]^{1/q} = a_k |x_k| \left( \sum_{n=k}^{\infty} A_n^{-q} \right)^{1/q}.
\]

On other hand, \( \sum_{k=1}^{\infty} f^{(k)} \) has a coordinate

\[
\sum_{k=1}^{\infty} f^{(k)} = \left( \sum_{k=1}^{\infty} f_1^{(k)}, \sum_{k=1}^{\infty} f_2^{(k)}, \ldots \right).
\]

So that

\[
\left\| \sum_{k=1}^{\infty} f^{(k)} \right\|_q = \left[ \sum_{n=1}^{\infty} \left( \sum_{k=1}^{\infty} f^{(k)} \right)^q \right]^{1/q} = \left[ \sum_{n=1}^{\infty} \left( \sum_{k=1}^{n} A_n^{-1} a_k x_k \right)^q \right]^{1/q} = \| Ax \|_q.
\]

We also have

\[
\left\| \sum_{k=1}^{\infty} f^{(k)} \right\|_q \geq \sum_{k=1}^{\infty} \| f^{(k)} \|_q.
\]

Thus,

\[
\| Ax \|_q \geq \sum_{k=1}^{\infty} a_k |x_k| \left( \sum_{n=k}^{\infty} A_n^{-q} \right)^{1/q} \geq \sup_{k \in \mathbb{N}} a_k \left( \sum_{n=k}^{\infty} A_n^{-q} \right)^{1/q} \sum_{k=1}^{\infty} |x_k|.
\]

Hence,

\[
\| A \|_{1,q} \geq \sup_{k \in \mathbb{N}} a_k \left( \sum_{n=k}^{\infty} A_n^{-q} \right)^{1/q}.
\]

From (3.1.7) and (3.1.8) we obtain (3.1.3).

**Theorem 3.1.1.** Let \( A \) be the weighted mean matrix given by (3.1.1).

(a) If \( 1 < p \leq q < \infty \) the following conditions are equivalent:

(i) \( A \) maps \( l^p \) into \( l^q \).
(ii) For some constant $K_1$ and all $m = 1, 2, \ldots$ such that
\[
\sum_{n=1}^{m} \left( A_n^{-1} \sum_{k=1}^{n} a_k^{q'} \right)^{q/p} \leq K_1 \left( \sum_{k=1}^{m} a_k^{q'} \right)^{q/p}.
\]

(iii) For some constant $K_2$ and all $m = 1, 2, \ldots$ such that
\[
\left( \sum_{n=m}^{\infty} A_n^{-q} \right)^{1/q} \left( \sum_{k=1}^{m} a_k^{p^*} \right)^{1/p^*} \leq K_2.
\]

(iv) For some constant $K_3$ and all $m = 1, 2, \ldots$ such that
\[
\sum_{k=m}^{\infty} \left( a_k \sum_{n=k}^{\infty} A_n^{-q} \right)^{p^*} \leq K_3 \left( \sum_{n=m}^{\infty} A_n^{-q} \right)^{p^*/q^*}.
\]

(b) If $1 \leq q < p < \infty$ the $A$ does not map $l^p$ into $l^q$.

Proof. Part (a) is special case of Theorem 2.1.1 to weighted mean matrix with $a_n$ is replaced by $A_n^{-1}$, $b_k$ by $a_k$. Part (b) is a consequence of Proposition 3.1.1 below. \qed

**Proposition 3.1.1.** Let $A$ be a lower triangular matrix with
\[
\rho = \lim_{n \to \infty} \inf \sum_{k=1}^{n} a_{nk} > 0.
\]
If $0 < q < p < \infty$ then $A$ is not bounded $l^p$ in $l^q$.

To prove the Proposition 3.1.1 we need to prove that Pitt’s Theorem [5].

**Theorem 3.1.2** (Pitt’s Theorem). If $1 \leq q < p < \infty$. Then every bounded linear operator from $l^q$ into $l^p$ is compact.

To prove Pitt’s Theorem we need to show the following Lemma that is proved in paper of D. Sylvain [5]

**Lemma 3.1.1.** If $z \in l^p$, $1 \leq p < \infty$ and for every weakly null sequence $(w^{(i)})$ in $l^p$, then
\[
\limsup_{i \to \infty} \| z + z^{(i)} \|_p^p = \| z \|_p^p + \limsup_{i \to \infty} \| z^{(i)} \|_p^p.
\]

Proof. Let $z^{(k)} = (z_1, z_2, \ldots, z_k, 0, \ldots)$ and $z^{(k)} \to z$ in $l^p$. We also have
\[
\| z^{(k)} - z \|_p = \left( \sum_{j=k+1}^{\infty} |z_j|^p \right)^{1/p}.
\]

First, we will prove that equality is true for $(z^{(k)})$. Since $(w^{(i)})$ is weakly null sequence so that
\[
w^{(i)}_n \to 0 \quad \text{as} \quad i \to \infty, \forall n.
\]
We have
\[
\limsup_{i \to \infty} \| w^{(i)} \|_p^p = \limsup_{i \to \infty} \left[ \sum_{j=1}^{k} |w_j^{(i)}|^p + \sum_{j=k+1}^{\infty} |w_j^{(i)}|^p \right] = \limsup_{i \to \infty} \sum_{j=k+1}^{\infty} |w_j^{(i)}|^p.
\]

Fixing \( k \), we have
\[
\limsup_{i \to \infty} \sum_{j=1}^{k} |z_j^{(k)} + w_j^{(i)}|^p = \sum_{j=1}^{k} |z_j^{(k)}|^p.
\]

Therefore,
\[
\limsup_{i \to \infty} \| z^{(k)} + z^{(i)} \|_p^p = \limsup_{i \to \infty} \sum_{j=1}^{\infty} |z_j^{(k)} + w_j^{(i)}|^p
\]
\[
= \limsup_{i \to \infty} \left[ \sum_{j=1}^{k} |z_j^{(k)} + w_j^{(i)}|^p + \sum_{j=k+1}^{\infty} |z_j^{(k)} + w_j^{(i)}|^p \right]
\]
\[
= \sum_{j=1}^{k} |z_j^{(k)}|^p + \limsup_{i \to \infty} \sum_{j=k+1}^{\infty} |w_j^{(i)}|^p
\]
\[
= \| z^{(k)} \|_p^p + \limsup_{i \to \infty} \| w^{(i)} \|_p^p.
\]

Next, we will prove that Lemma 3.1.1 is true for \( z \in l^p \). We use the inequality
\[
\| x \|_p^p - \| y \|_p^p \leq p \| x - y \|_p \left( \| x \|_p + \| y \|_p \right)^{p-1}.
\]

Thus,
\[
\| z + w^{(i)} \|_p^p - \| z^{(k)} + w^{(i)} \|_p^p \leq p \| z - z^{(k)} \|_p \left( \| z + w^{(i)} \|_p + \| z^{(k)} + w^{(i)} \|_p \right)^{p-1}
\]
\[
\leq p \| z - z^{(k)} \|_p \left( 2 \| z \|_p + 2 \| w^{(i)} \|_p \right)^{p-1}
\]
\[
\leq p C_p \| z - z^{(k)} \|_p.
\]

Hence,
\[
\| z^{(k)} + w^{(i)} \|_p^p - p C_p \| z - z^{(k)} \|_p \leq \| z + w^{(i)} \|_p^p \leq \| z^{(k)} + w^{(i)} \|_p^p + p C_p \| z - z^{(k)} \|_p.
\]

That means
\[
\| z^{(k)} \|_p^p + \limsup_{i \to \infty} \| w^{(i)} \|_p^p - p C_p \| z - z^{(k)} \|_p \leq \| z + w^{(i)} \|_p^p \leq \| z^{(k)} \|_p^p + \limsup_{i \to \infty} \| w^{(i)} \|_p^p + p C_p \| z - z^{(k)} \|_p.
\]

Since \( \| z - z^{(k)} \|_p \to 0 \), then
\[
\| z^{(k)} \|_p^p + \limsup_{i \to \infty} \| w^{(i)} \|_p^p = \| z + w^{(i)} \|_p^p.
\]

Hence, we complete Lemma 3.1.1. 

□
Now, we come back Pitt’s Theorem that is shown in the paper of D. Sylvain [5].

**Proof of Pitt’s Theorem.** Let \( T : l^q \rightarrow l^p \) be a norm-one operator. Every bounded sequence in \( l^p \) has a weakly Cauchy sequence. Thus, for proving the compactness of \( T \), it is enough to show that \( T \) is weak-to-norm continuous. So, let us consider a weakly sequence \( (h_n) \) in \( l^q \). We have to show that

\[
\lim_{n \to \infty} \| T(h_n) \| = 0.
\]

Fix \( 0 < \varepsilon < 1 \). By definition of the norm of \( T \), we have \( \| T \|_{p,q} = 1 \) and there exists \( x_\varepsilon \in l^p \) such that \( \| x_\varepsilon \|_p = 1 \) and \( 1 - \varepsilon \leq \| T(x_\varepsilon) \| \leq 1 \). Moreover, for all \( n \in \mathbb{N} \) and for all \( t > 0 \)

\[
\| T(x_\varepsilon) + T(th_n) \|_q \leq \| T \|_{p,q} \| x_\varepsilon \|_p + \| th_n \|_p = \| x_\varepsilon \|_p + \| th_n \|_p.
\]

(3.1.11)

In the left-hand side of (3.1.11), we apply Lemma 3.1.1 in \( l^q \), with \( z = T(x_\varepsilon) \) and the weakly null sequence \( (th_n) \). We have

\[
\limsup_{n \to \infty} \| T(x_\varepsilon) + T(th_n) \|_q = \| T(x_\varepsilon) \|_q + \limsup_{n \to \infty} \| T(th_n) \|_q.
\]

Similarly, we also apply Lemma 3.1.1 to the right-hand side of (3.1.11) with \( z = x_\varepsilon \) and the weakly null sequence \( (th_n) \) to obtain

\[
\limsup_{n \to \infty} \| x_\varepsilon + th_n \|_p = \| x_\varepsilon \|_p + \limsup_{n \to \infty} \| th_n \|_p.
\]

Thus,

\[
\left[ \| T(x_\varepsilon) \|_q^q + t^q \limsup_{n \to \infty} \| Th_n \|_q \right]^{1/q} = \left[ \limsup_{n \to \infty} \| T(x_\varepsilon) + T(th_n) \|_q \right]^{1/q} \leq \left[ \limsup_{n \to \infty} \| x_\varepsilon + th_n \|_p \right]^{1/p} = \left[ \| x_\varepsilon \|_p^p + \limsup_{n \to \infty} \| th_n \|_p \right]^{1/p} = \left[ \| x_\varepsilon \|_p^p + t^p \limsup_{n \to \infty} \| h_n \|_p \right]^{1/p}.
\]

Recall that \( \| x_\varepsilon \| = 1, 1 - \varepsilon \leq \| T(x_\varepsilon) \| \leq 1 \) and that \( (h_n) \) is weakly convergent so it is bounded by some \( M > 0 \). This gives

\[
\limsup_{n \to \infty} \| Th_n \|_q^q \leq \frac{1}{t^q} \left[ (1 + t^p M^p)^q - (1 - \varepsilon)^q \right].
\]

Taking \( t = \varepsilon^{1/p} \) here, we get

\[
\limsup_{n \to \infty} \| Th_n \|_q^q \leq \frac{1}{\varepsilon^{q/p}} \left[ (1 + \varepsilon M^p)^{q/p} - (1 - \varepsilon)^q \right] \leq \frac{1}{\varepsilon^{q/p}} \left[ 1 + \frac{q}{p} M^p \varepsilon - (1 - q\varepsilon) \right].
\]

Because \( (1 - \varepsilon)^q \leq 1 - q\varepsilon \) with \( 0 \leq \varepsilon < 1 \) and \( q \geq 1 \), Letting \( \varepsilon \to 0 \) we obtain

\[
\limsup_{n \to \infty} \| Th_n \|_q^q \leq 0,
\]

and therefore the sequence \( (T(h_n)) \) norm-converges to 0. \( \square \)
We come back Proposition 3.1.1.

**Proof of Proposition 3.1.1.** We assume that $A$ is satisfied the hypothesis of Proposition 3.1.1 and $A$ is bounded from $l^p$ to $l^q$. By Pitt’s Theorem, we have $A$ must be compact operator. On the other hand, $ho = \liminf_{n \to \infty} | \sum_{k=1}^{n} a_{nk} | > 0$.

So there exists $N_0$, for all $n \geq N_0$

$$\left| \sum_{k=1}^{n} a_{nk} \right| > \frac{\rho}{2^{1/(2q)}}.$$

For $m > N_0$, we have

$$\sum_{n=1}^{m} \left| \sum_{k=1}^{n} a_{nk} \right|^q > (m - N_0) \frac{\rho^q}{\sqrt{2}}.$$

Hence, we can choose $m > 4N_0$ to

$$\sum_{n=1}^{m} \left| \sum_{k=1}^{n} a_{nk} \right|^q > \frac{3m \rho^q}{4 \sqrt{2}} > \frac{\rho^q m}{2}.$$  (3.1.12)

Let $x^{(n)} \in l^q$ such that

$$x^{(n)}_k = \begin{cases} n^{-1/q} & \text{if } 1 \leq k \leq n, \\ 0 & \text{if } k > n. \end{cases}$$  (3.1.13)

We have to point out $x^{(n)} \to 0$ in weakly in $l^q$ that means

$$| < x^{(n)}, y > | = n^{-1/p} \sum_{k=1}^{n} |y_k| \to 0 \text{ as } n \to \infty.$$  

where $y \in l^p$. Since $y \in l^p$ so $\sum_{k=1}^{\infty} |y_k|^q < +\infty$. Thus,

$$\lim_{n \to \infty} \sum_{k=n+1}^{\infty} |y_k|^p = 0.$$

Let $\epsilon > 0, \exists k_0, \forall n \geq k_0,$

$$\sum_{k=n+1}^{\infty} |y_k|^q < \left( \frac{\epsilon}{2} \right)^q.$$

So we get

$$\sum_{k=k_0+1}^{\infty} |y_k|^q < \left( \frac{\epsilon}{2} \right)^q.$$

For $n > k_0$, 

$$\sum_{k=k_0+1}^{n} |y_k| \leq \left( \sum_{k=k_0+1}^{n} 1^q \right)^{1/q} \left( \sum_{k=k_0+1}^{n} |y_k|^q \right)^{1/q} < n^{1/p} \frac{\epsilon}{2}.$$
Hence,
\[ n^{-1/p} \sum_{k=1}^{n} |y_k| = \frac{\sum_{k=1}^{k_0} |y_k| + \sum_{k=k_0+1}^{n} |y_k|}{n^{1/p}} \]
\[ \leq \frac{\sum_{k=1}^{k_0} |y_k|}{n^{1/p}} + \frac{\epsilon}{2} \]
\[ = C \frac{1}{n^{1/p}} + \frac{\epsilon}{2}. \]

We also have
\[ \lim_{n \to \infty} \frac{1}{n^{1/p}} = 0. \]

Then for \( \epsilon \) is chosen above, \( \exists n_0 : \forall n \geq n_0 \)
\[ \frac{1}{n^{1/p}} < \frac{\epsilon}{2C}. \]

Therefore,
\[ n^{-1/p} \sum_{k=1}^{n} |y_k| < \epsilon. \]

We have shown \( x^{(n)} \to 0 \) is weakly in \( l^q \). Since \( A \) is compact so that
\[ \|Ax^{(n)}\|_q \to 0. \]

But if \( m \geq 4N_0 \), we have
\[ \|Ax^{(m)}\|_q \geq \sum_{n=\varepsilon}^{m} \left| \sum_{k=1}^{n} a_{nk}x^{(m)}_k \right|^q \]
\[ = \sum_{n=1}^{m} m^{-1} \left| \sum_{k=1}^{n} a_{nk} \right|^q \]
\[ \geq \frac{\rho^q}{2}. \]

This is contradiction (3.1.12) as \( m \to \infty \) \( \square \)

Theorem 3.1.1 seems to new event in the special \( q = p \). The best previously known results in this case are given in an unpublished Thesis of Jame Cartlidge. Cartlidge’s main result from which he derives many interesting corollaries, is the following

**Theorem 3.1.3** (Theorem C). Let \( p \) be the fixed , \( 1 < p < \infty \). Let \( A \) be given by (3.1.1) and suppose that \( a_n > 0 \) for \( n = 1, 2, \ldots \) If
\[ L = \sup_n \left( \frac{A_{n+1}}{a_{n+1}} - \frac{A_n}{a_n} \right) < p, \]
then \( A \) is bounded on \( l^p \) and
\[ \|A\|_{p,q} \leq \frac{p}{p-L}. \]
**Proof.** We use the proof of P. Gao [3] to show that the Theorem C. To prove the theorem we will show that

\[
\sum_{n=1}^{\infty} B_p^p \leq \frac{p}{p-L} \sum_{n=1}^{\infty} a_n B_{n-1}^p
\]  

(3.1.15)

where

\[
B_n = \sum_{k=1}^{n} \frac{a_k x_k}{A_n}.
\]

We consider the function

\[
f(x_n) = \left( \frac{A_n}{a_n} + p - 1 \right) B_p^p - p x_n B_{n-1}^p.
\]

For \( n \geq 2, x_n \geq 0 \), we have

\[
f'(x_n) = p(p-1) \frac{a_n}{A_n} (B_{n-1}^p - x_n B_{n-2}^p).
\]

Hence \( f'(x_n) = 0 \) where \( B_n = x_n \). That means

\[
\frac{\sum_{k=1}^{n-1} a_k x_k + a_n x_n}{A_n} = a_n.
\]

or \( B_{n-1} = x_n \). We also have

\[
f(x_n) = B_{n-1}^p \left[ \left( \frac{A_n}{a_n} + p - 1 \right) B_n - p x_n \right]
\]

\[
= B_{n-1}^p \left[ \left( \frac{A_n}{a_n} + p - 1 \right) \sum_{k=1}^{n-1} a_k x_k + \left( \frac{A_n}{a_n} + p - 1 \right) \frac{a_n x_n}{A_n} - p x_n \right]
\]

\[
= B_{n-1}^p \left[ \left( \frac{A_n}{a_n} + p - 1 \right) \sum_{k=1}^{n-1} a_k x_k + x_n (p-1) \left( \frac{n}{A_n} - 1 \right) \right].
\]

Since \( a_n/A_n \leq 1 \) so that

\[
\lim_{x_n \to \infty} f(x_n) \leq 0.
\]

That means \( \exists y \) such that \( f(y) < M \) and \( \exists x \in [B_{n-1}, y] \) satisfied

\[
f'(x) = \frac{f(y) - f(B_{n-1})}{y - B_{n-1}} < 0.
\]

Similarly, we have

\[
f(0) = \left( \frac{A_n}{a_n} + p - 1 \right) \left[ \sum_{k=1}^{n-1} a_k x_k \frac{A_n}{A_{n-1}} \right]^p \geq 0.
\]

Thus \( f(x_n) \leq f(B_{n-1}) \). Therefore

\[
\left( \frac{A_n}{a_n} + p - 1 \right) B_p^p - p x_n B_{n-1}^p \leq \left( \frac{a_n}{A_n} - 1 \right) B_{n-1}^p.
\]

(3.1.16)
CHAPTER 3. WEIGHTED MEAN MATRICES

By defining $B_0 = 0$ the above inequality also holds for $n = 1$. Summing (3.1.16) from $n = 1$ to $N$ gives

$$\sum_{n=1}^{N} \left( \frac{A_n}{a_n} + p - 1 \right) B_n^p - p \sum_{n=1}^{N} x_n B_{n-1}^p \leq \sum_{n=1}^{N} \left( \frac{a_n}{A_n} - 1 \right) B_{n-1}^p.$$  

So

$$\sum_{n=1}^{N} \left( \frac{A_n}{a_n} + p - 1 \right) B_n^p - \sum_{n=0}^{N-1} \left( \frac{a_{n+1}}{A_{n+1}} - 1 \right) B_n^p \leq p \sum_{n=1}^{N} x_n B_{n-1}^p.$$  

Hence

$$\sum_{n=1}^{N} \left( \frac{A_n}{a_n} + p - \frac{A_{n+1}}{a_{n+1}} \right) B_n^p + p \sum_{n=1}^{N} \left( 1 + \frac{A_{n+1}}{a_{n+1}} \right) B_n^p - \sum_{n=0}^{N-1} \left( \frac{a_{n+1}}{A_{n+1}} - 1 \right) B_n^p \leq p \sum_{n=1}^{N} x_n B_{n-1}^p.$$  

(3.1.17)

By condition (3.1.14) we have

$$p + \frac{A_n}{a_n} - \frac{A_{n+1}}{a_{n+1}} \geq p - L.$$  

We get the inequality (3.1.15) follows (3.1.17). Apply the Holder inequality we have

$$\sum_{n=1}^{N} x_n B_{n-1}^p \leq \left( \sum_{n=1}^{N} x_n^p \right)^{p/(p-1)} \left( \sum_{n=1}^{N} B_n^p \right)^{(p-1)/p}.$$  

(3.1.18)

Combining (3.1.15) and (3.1.18) we obtain

$$\left( \sum_{n=1}^{N} B_n^p \right)^{1/p} \leq \frac{p}{p - L} \left( \sum_{n=1}^{N} x_n^p \right)^{1/p}.$$  

This completes the proof.

Cartlidge’s result is less satisfactory than Theorem 3.1.1 since his hypothesis (3.1.14) is highly susceptible to changes individual $a_n$ s. On the other hand, Theorem 3.1.3 cover many case of partial interest, wherein the $a_n$ s be have in a ”regular” fashion. A particularly interesting consequence of Theorem 3.1.3, again due to Cartlidge is

**Corollary 3.1.1** (Corollary C). If $(a_n)_{n=1}^{\infty}$ is an increasing sequence then $A$ is bounded on $l^p$ where $1 < p < \infty$.

**Proof.** Since $(a_n)_{n=1}^{\infty}$ is an increasing sequence so that

$$\frac{A_{n+1}}{a_{n+1}} - \frac{A_n}{a_n} \leq \frac{A_{n+1}}{a_n} - \frac{A_n}{a_n} = 1.$$  

Apply Theorem (3.1.3), we get the proof.
3.2 Some Corollaries of Theorem 3.1.1

**Corollary 3.2.1.** Let \( p \) be the fixed, \( 1 < p < \infty \) if

\[
a_1^p + \cdots + a_n^p = O\left(\frac{1}{n^{1/(p-1)}} A_n\right),
\]

then \( A \) is bounded on \( l^p \).

*Proof.* From the hypothesis \( (3.2.1) \) we get

\[
\frac{a_1^p + \cdots + a_n^p}{A_n^{1/(p-1)}} \leq K,
\]

then

\[
\left(\sum_{k=1}^{n} a_k^p\right)^p A_n^{-p} \leq K^p a_n^p.
\]

So that

\[
\sum_{n=1}^{m} \left(\frac{1}{n^{1/(p-1)}} \sum_{k=1}^{n} a_k^p\right)^p \leq K^p \sum_{n=1}^{m} a_n^p.
\]

The above inequality is a part \((ii)\) of Theorem 3.1.1 with \( p = q \) then \( A \) is bounded on \( l^p \). \( \square \)

**Corollary 3.2.2.** Let \( p \) be the fixed and \( 1 < p < \infty \). If

\[
a_n \sum_{k=n}^{\infty} A_k^{-p} = O(A_n^{1-p}),
\]

then \( A \) is bounded on \( l^p \).

*Proof.* By the hypothesis we have

\[
a_n \sum_{k=n}^{\infty} A_k^{-p} < K A_n^{1-p}.
\]

So that

\[
a_n^p \left(\sum_{k=n}^{\infty} A_k^{-p}\right)^p < K^p A_n^{-p}.
\]

Thus,

\[
\sum_{n=m}^{\infty} \left(a_n \sum_{k=n}^{\infty} A_k^{-p}\right)^p < K^p \sum_{n=m}^{\infty} A_n^{-p}.
\]

By the part \((iv) \Rightarrow (i)\) of Theorem 3.1.1 with \( p = q \) we point out \( A \) is bounded on \( l^p \). \( \square \)

After that, we give a counterexample to point out the inverse direction of Corollary 3.2.1 does not hold that means \( A \) is bounded but

\[
\limsup_{n \to \infty} \frac{\sum_{j=1}^{n} a_j^{p^*}}{\frac{1}{n^{1/(p-1)}} A_n} = \infty.
\]

(3.2.2)
Let

\[ a_n = \begin{cases} 
1 & \text{if } n \neq 2^k, \\
2^{-k} & \text{if } n = 2^k.
\end{cases} \]

For \(2^k \leq n < 2^{k+1}\), we have

\[ A_n = \sum_{j=1}^{n} a_j = \sum_{r=0}^{k} 2^{-r} + n - k = 2 - \frac{1}{2^k} + n - k, \]

\[ \sum_{j=1}^{n} a_j^{p^*} = \sum_{r=0}^{k} 2^{-rp^*} + n - k = \frac{2^{p^*}}{2^{p^*} - 1} \left[ 1 - \frac{1}{2^{(k+1)p^*}} \right] + n - k. \]

We compute

\[ I = \sum_{j=1}^{2^k} a_j^{p^*} \cdot 2^{1/(p-1)} A_{2^k} > \frac{2^k - k}{2^{-k/(p-1)}(2^k + 2 - k)} = \frac{(2^k - k)2^{k/(p-1)}}{2^k + 2 - k}. \]

Thus \(k \to \infty\), we get \(I \to \infty\). Therefore, we have point out (3.2.2).

Next, we will check the boundedness of \(A\). By Corollary 3.2.2 we need only to show that

\[ a_m \sum_{n=m}^{\infty} A_n^{-p} = O(A_m^{1-p}). \]

We have

\[ A_n = 2 - \frac{1}{2^k} + n - k \geq 2 - \frac{2}{n} + n - \log_2 n \geq \frac{n}{2}. \]

So

\[ A_n^{-p} \leq \frac{2^p}{n^p}. \]

Thus,

\[ A_m^{1-p} a_m \sum_{n=m}^{\infty} A_n^{-p} \leq m^{p-1} \sum_{n=m}^{\infty} \frac{2^p}{n^p} \sim m^{p-1} 2^p m^{1-p} = 2^p. \]

Therefore, \(A\) is bounded.

We continue to give an example to point out the inverse direction of Corollary 3.2.2 is not true that means \(A\) is bounded but

\[ \limsup_{m \to \infty} a_m \sum_{n=m}^{\infty} A_n^{-p} = +\infty. \] (3.2.3)

Taking

\[ a_n = \begin{cases} 
1 & \text{if } n \neq 2^k, \\
k & \text{if } n = 2^k.
\end{cases} \] (3.2.4)

For \(2^k \leq n < 2^{k+1}\), we have

\[ A_n = \sum_{j=1}^{2^k} a_j + \sum_{j=2^k+1}^{n} a_j = \sum_{r=0}^{k} r + (2^k - k) + (n - 2^k) = \frac{k(k + 1)}{2} + n - k, \]
\[
\sum_{j=1}^{n} a_{j}^{p^*} = \sum_{r=0}^{k} r^{p} + n - k \leq k^{p^*+1} + n - k.
\]

We have
\[
A_n = \frac{k(k+1)}{2} + n - k \geq \frac{(\log_2 n - 1) \log_2 n}{2} + n - \log_2 n \geq \frac{n}{2},
\]
\[
A_n = \frac{k(k+1)}{2} + n - k \leq \frac{\log_2 n (\log_2 n + 1)}{2} + n \leq 2n.
\]

We estimate
\[
A_{2k}^{p-1} a_{2k}^{p} \sum_{n=2k}^{\infty} A_n^{-p} \geq \left(\frac{2^k}{2}\right)^{p-1} k \sum_{n=2k}^{\infty} \frac{1}{(2n)^p} \sim k \frac{2^k}{2^{p-1} 2^{k(p-1)} 2^{k(1-p)}} = \frac{k}{2^{2p-1}}.
\]

Hence, we get (3.2.3).

To check that the boundedness of A, we use Corollary 3.2.1. Thus, we need only to show that
\[
\sum_{j=1}^{n} a_{j}^{p^*} = O(a_n^{1/(p-1)} A_n).
\]

We have
\[
\frac{\sum_{j=1}^{n} a_{j}^{p^*}}{a_n^{1/(p-1)} A_n} \leq \frac{k^{p^*+1} + n - k}{k(k+1)/2 + n - k} \leq \frac{(\log_2 n)^{p^*+1} + n}{n/2} = \frac{2(\log_2 n)^{p^*+1}}{n} + 1/2.
\]

We have
\[
\lim_{n \to \infty} \frac{2(\log_2 n)^{p^*+1}}{n} = 0.
\]

Thus, there exists \( C > 0 \) such that
\[
\frac{2(\log_2 n)^{p^*+1}}{n} < C.
\]

Therefore,
\[
\frac{\sum_{j=1}^{n} a_{j}^{p^*}}{a_n^{1/(p-1)} A_n} \leq C + \frac{1}{2}.
\]

So that A is bounded.
Now, we give another sequence $a_n$ such that $A$ is bounded but $a_n$ is not satisfied
\[a_m \sum_{n=m}^{\infty} A_n^{-p} = O(A_m^{1-p}) \quad \text{and} \quad \sum_{j=1}^{n} a_j^* = O(a_n^{1/(p-1)} A_n).\]

Let
\[a_n = \begin{cases} 1 & \text{if } n \neq 2^k, \\ 2^{-k} & \text{if } n = 2^k, \text{ } k \text{ is odd}, \\ k & \text{if } n = 2^k, \text{ } k \text{ is even}. \end{cases}\]

For $2^k \leq n < 2^{k+1}$, we get
\[A_n = \sum_{j=0}^{2^k} a_j + n - 2^k.\]

If $k = 2m$, we have
\[A_{2^k} = A_{2^{2m}} = \sum_{r=0}^{2m} a_{2r} + 2^{2m} - 2m = \sum_{i=0}^{m-1} a_{2^{2i}+1} + \sum_{i=1}^{m} a_{2^{2i}} + 2^{2m} - 2m = \sum_{i=0}^{m-1} (2i + 1) + \sum_{i=1}^{m} 2^{-2i} + 2^{2m} - 2m = m(m-1) + m + \frac{1}{3} \left(1 - \frac{1}{4^m}\right) + 2^{2m} - 2m = m^2 + 2^{2m} + \frac{1}{3} \left(1 - \frac{1}{4^m}\right) - 2m.\]

Similarly, if $k = 2m + 1$, we also have
\[A_{2^k} = \sum_{i=0}^{m} (2i + 1) + \sum_{i=1}^{m} 2^{-2i} + 2^{2m+1} - 2m - 1 = m(m+1) + m + \frac{1}{3} \left(1 - \frac{1}{4^m}\right) + 2^{2m+1} - 2m - 1 = m^2 + 2^{2m+1} + \frac{1}{3} \left(1 - \frac{1}{4^m}\right) - 1.\]

Thus,
\[A_n = \begin{cases} m^2 + \frac{1}{3} \left(1 - \frac{1}{4^m}\right) - 2m + n & \text{if } n = 2^{2m}, \\ m^2 + \frac{1}{3} \left(1 - \frac{1}{4^m}\right) - 1 + n & \text{if } n = 2^{2m+1}. \end{cases}\]  \hspace{1cm} (3.2.5)

We can see that
\[2n > A_n \geq A_{2^k} > 2^k > \frac{n}{2}.\]
If $k$ is even, we have
\[
A_{2m}^{p-1} \sum_{n=2m}^{\infty} A_n^{-p} \geq (2^{2m} - 2m)^{p-1} 2m \sum_{n=2m}^{\infty} \left( \frac{1}{2n} \right)^p \\
\geq (2^{2m} - 2m)^{p-1} 2m \frac{2^{2m(1-p)}}{2p} \\
\geq \frac{m}{2^{p-1}} \left[ 1 - \frac{2m}{2^{2m}} \right].
\]
So the left hand side tends to infinite where $m$ tends to infinite.

If $k$ is odd, we have
\[
\sum_{j=1}^{2^{2m+1}} a_j^{p^*} \\frac{a_j^{1/(p-1)}}{A_{2m+1}^{1/(p-1)}} \geq \frac{2^{2m+1} 2(2m+1)/(p-1)}{2(m^2 + 2^{2m+1} + \frac{1}{3})} \\
= \frac{2(2m+1)^{p^*}}{2(m^2 + 2^{2m+1} + \frac{1}{3})}
\]
We see that the left hand side is also tends to infinite as $m$ to infinite. We check the boundedness of $A$
\[
\sum_{n=1}^{l} \left( A_n^{-1} \sum_{j=1}^{n} a_j^{p^*} \right)^p \leq K \sum_{n=1}^{l} a_n^{p^*}.
\]
For $2^s \leq l < 2^{s+1}$, we have
\[
\sum_{n=1}^{l} a_n^{p^*} \geq \sum_{n=1}^{2^s} a_n^{p^*} \geq 2^s - s.
\]
On other hand,
\[
\sum_{j=1}^{n} a_j^{p^*} \leq \sum_{j=1}^{2^{k+1}-1} a_j^{p^*} \leq \sum_{r=0}^{k+1} r^{p^*} + 2^{k+1} - k \leq (k+1)^{p^*+1} + 2^{k+1}.
\]
Then
\[
\sum_{n=1}^{l} \left( A_n^{-1} \sum_{j=1}^{n} a_j^{p^*} \right)^p \leq \sum_{n=1}^{2^{k+1}-1} \left( A_n^{-1} \sum_{j=1}^{n} a_j^{p^*} \right)^p \\
= \sum_{k=0}^{s} \sum_{2^k \leq n < 2^{k+1}} \left( A_n^{-1} \sum_{j=1}^{n} a_j^{p^*} \right)^p \\
\leq \sum_{k=0}^{s} 2^k \left( \frac{(k+1)^{p^*+1} + 2^{k+1}}{2^k} \right)^p.
\]
Since
\[
\lim_{k \to \infty} \frac{(k+1)^{p^*+1}}{2^k} = 0,
\]
then there exists $K > 0$ such that
\[
\frac{(k+1)^{p^*+1} + 1}{2^k} < K.
\]
Hence,
\[
\sum_{n=1}^{l} \left( A_n^{-1} \sum_{j=1}^{n} a_j^p \right)^p \leq \sum_{k=0}^{s} 2^k (K + 2)^p = (K + 2)^{p2^{s+1}}. \tag{3.2.7}
\]

From (3.2.6) and (3.2.7), we get
\[
\sum_{n=1}^{l} \left( A_n^{-1} \sum_{j=1}^{n} a_j^p \right)^p \leq \frac{(K + 2)^{p2^{s+1}}}{2^s - s} \leq 4(K + 2)^p.
\]

We deduce A is bounded.

**Corollary 3.2.3.** Let \( p \) be the fixed, \( 1 < p < \infty \). If A is bounded on \( l^p \), then
\[
\sum_{k=n}^{\infty} A_k^{-p} = O(nA_n^{-p}). \tag{3.2.8}
\]

**Proof.** Since A is bounded on \( l^p \) so that we apply Theorem 3.1.1 (a): \( (i) \Rightarrow (iii) \) with \( p = q \) we obtain
\[
\left( \sum_{k=n}^{\infty} A_k^{-p} \right)^{1/p} \left( \sum_{m=1}^{n} a_m^p \right)^{1/p^*} \leq K.
\]

That means
\[
\sum_{k=n}^{\infty} A_k^{-p} \left( \sum_{m=1}^{n} a_m^p \right)^{p/p^*} \leq K^p. \tag{3.2.9}
\]

Using the Holder’s inequality we also have
\[
\sum_{m=1}^{n} a_m \leq n^{1/p} \left( \sum_{m=1}^{n} a_m^p \right)^{1/p^*}. \tag{3.2.10}
\]

By the inequalities (3.2.9) and (3.2.10) we get
\[
\frac{1}{n} \sum_{k=n}^{\infty} A_k^{-p} \left( \sum_{m=1}^{n} a_m \right)^p \leq K^p.
\]

We deduce
\[
\sum_{k=n}^{\infty} A_k^{-p} \leq K^p nA_n^{-p}.
\]

The question: Is A bounded on \( l^p \) when inequality (3.2.8) holds? We give the example to point out the answer of the above question is no. We choose
\[
a_k = \begin{cases} 
    k & \text{when } m \in \mathbb{Z}_+ \text{ for } k = 2^m, \\
    0 & \text{otherwise}.
\end{cases} \tag{3.2.11}
\]
For $2^k \leq n < 2^{k+1}$ and let $a_0 = 1$ then

$$A_n = 1 + \sum_{j=1}^{n} a_j = 1 + \sum_{l=0}^{k} 2^l = 2^{k+1},$$

$$\sum_{j=0}^{n} a_j^p = 1 + \sum_{l=0}^{k} 2^{lp} = 1 + \frac{2^{(k+1)p^*} - 1}{2^{p*} - 1}.$$ 

We calculate

$$\sum_{j=n}^{\infty} A_j^{-p} = \sum_{j=n}^{2^{k+1}-1} A_j^{-p} + \sum_{j=2^{k+1}}^{\infty} A_j^{-p}$$

$$= \sum_{j=n}^{2^{k+1}-1} 2^{-(k+1)p} + \sum_{l=k+1}^{\infty} \sum_{2^l \leq k < 2^{l+1}} 2^{-(l+1)p}$$

$$= 2^{-(k+1)p} (2^{k+1} - n) + \sum_{l=k+1}^{\infty} \sum_{2^l \leq k < 2^{l+1}} 2^{-(l+1)p}$$

$$= 2^{-(k+1)p} (2^{k+1} - n) + \sum_{l=k+1}^{\infty} 2^{l-2(l+1)p}$$

$$= 2^{-(k+1)p} (2^{k+1} - n) + \frac{2^{-p} 2^{(k+1)(1-p)}}{1 - 2^{1-p}}$$

$$= 2^{-(k+1)p} (2^{k+1} - n) + \frac{2^{(k+1)(1-p)}}{2p - 2}$$

$$= \frac{[(2^{k+1} - n)(2^p - 2) + 2^{k+1}] 2^{-(k+1)p}}{2p - 2}.$$ 

We have

$$\frac{\sum_{j=n}^{\infty} A_j^{-p}}{n A_n^{-p}} = \frac{(2^{k+1} - n)(2^p - 2) + 2^{k+1}}{n(2^p - 2)} < \frac{2^p}{2p - 2}.$$ 

We need only to check that

$$\sup_{n} \left( \left( \sum_{j=n}^{\infty} A_j^{-p} \right)^{1/p} \left( \sum_{j=0}^{n} a_j^{p*} \right)^{1/p^*} \right) = \infty.$$ 

$$\left( \sum_{j=n}^{\infty} A_j^{-p} \right)^{1/p} \left( \sum_{j=0}^{n} a_j^{p*} \right)^{1/p^*}$$

$$= \left( \frac{[(2^{k+1} - n)(2^p - 2) + 2^{k+1}] 2^{-(k+1)p}}{2p - 2} \right)^{1/p} \left( 1 + \frac{2^{(k+1)p^*} - 2^{p^*}}{2^{p*} - 1} \right)^{1/p^*}$$

$$> \left( \frac{[(2^{k+1} - n)(2^p - 2) + 2^{k+1}] 2^{-(k+1)p}}{2} \right)^{1/p} \frac{2^{(k+1)p^*}}{2}$$

$$> \frac{n^{1/p}}{8}.$$

We have shown the inverse of Corollary 3.2.3 is not true.
Chapter 3. Weighted Mean Matrices

Proposition 3.2.1. If $A$ is bounded on $l^p$ for some $1 \leq p < \infty$, then

$$\sum_{k=1}^{n} \frac{A_k}{k} = O(A_n).$$

Proof. We fix $n$ and define $x \in l^p$ by

$$x_k = \begin{cases} a_k^{1/p} & \text{for } k = 1, \ldots, n, \\ 0 & \text{otherwise} \end{cases}.$$  

We use the Holder’s inequality

$$A_k \leq k^{1/(p+1)} \left( \sum_{j=1}^{k} a_j^{1+1/p} \right)^{p/(p+1)}.$$

We see that $A$ is bounded on $l^p$, then

$$\sum_{k=1}^{n} \frac{A_k}{k} = \sum_{k=1}^{n} A_k^{-p} \frac{A_k^{p+1}}{k}$$

$$\leq \sum_{k=1}^{\infty} A_k^{-p} \left( \sum_{j=1}^{k} a_j^{1+1/p} \right)^p$$

$$= \|Ax\|_p^p$$

$$\leq \|A\|_{p,p}^p A_n.$$ 

\qed
Chapter 4

Littlewood’s problem

4.1 Littlewood’s problem

In this section we study a class of inequalities formulated by Littlewood in connection with some work on the general theory of orthogonal series. This simplest (non-trivial) examples are the inequality

\[ \sum_{n=1}^{\infty} a_n^3 \sum_{m=1}^{n} a_m^2 A_m \leq K \sum_{n=1}^{\infty} a_n^4 A_n^2 \]  
(4.1.1)

and a companion result

\[ \sum_{n=1}^{\infty} a_n A_n^2 \left( \sum_{m=n}^{\infty} a_m^{3/2} \right)^2 \leq K \sum_{n=1}^{\infty} a_n^2 A_n^4. \]  
(4.1.2)

We recall that \( a' \)'s are arbitrary non-negative numbers with partial sums \( A_n = a_1 + \cdots + a_n \) and \( K \) is an absolute constant. Now we find the constant \( K \) of (4.1.1). The special case of (4.1.1), in which \( a' \)'s are decreasing, is worth considering separately for then a particularly simple and natural proof is available.

We suppose that we have shown that the inequality (4.1.1) is true for decreasing sequence. We assume that \( a = (a_1, \ldots, a_i, a_{i+1}, \ldots) \) is new sequence that is not decreasing that means there exists \( i \in \mathbb{N} \) such that \( a_i < a_{i+1} \). We rearrange a new sequence \( a' = (a_1, \ldots, a_{i+1}, a_i, \ldots) \). We calculate

\[
\sum_{n=1}^{\infty} a_n^3 \sum_{m=1}^{n} a_m^2 A_m - \sum_{n=1}^{\infty} a_n^3 \sum_{m=1}^{n} a_m^2 A_m' \\
= a_i^3 \sum_{m=1}^{i} a_m^2 A_m + a_{i+1}^3 \sum_{m=1}^{i+1} a_m^2 A_m - a_i^3 \sum_{m=1}^{i+1} a_m^2 A_m' - a_{i+1}^3 \sum_{m=1}^{i+1} a_m^2 A_m' \\
= a_i^5 A_i + a_i^3 a_{i+1}^2 A_i + a_{i+1}^5 A_{i+1} - a_i^5 (A_i + a_{i+1} - a_i) - a_i a_{i+1}^2 (A_i + a_{i+1} - a_i) - a_{i+1}^5 A_{i+1} \\
= (a_i^5 - a_{i+1}^5) (A_i - A_{i+1}) + a_{i+1}^2 a_i^2 A_i (a_{i+1} - a_i) - a_i^3 a_{i+1}^2 (a_{i+1} - a_i) - a_{i+1}^5 (a_{i+1} - a_i) \\
= (a_i - a_{i+1}) \left[ a_{i+1} (a_i^4 + a_{i+1}^2 a_i + a_{i+1}^2 a_i^2 + a_i a_{i+1}^3 + a_i^4) + a_i^2 a_{i+1}^2 A_i - a_{i+1}^5 (a_{i+1} - a_i) \right] \\
= (a_i - a_{i+1}) \left[ a_{i+1}^4 a_i + a_{i+1}^3 a_i^2 + a_{i+1} a_i^4 + a_{i+1} a_i^2 A_i \right] > 0.
\]
We also have
\[
\sum_{n=1}^{\infty} a_n^4 A_n^2 - \sum_{n=1}^{\infty} a_n^4 A_n^2 \\
= a_i^4 A_i^2 + a_{i+1}^4 A_{i+1}^2 - a_i^4 (A_i + a_i - a_i)^2 - a_i^4 A_{i+1}^2 \\
= (a_i^4 - a_i^4)(A_i^2 - A_i^2) - 2a_i^4 A_i(a_i - a_i) - a_{i+1}^4 (a_{i+1} - a_i)^2 \\
= (a_{i+1} - a_i) [(a_{i+1}^2 + a_{i+1}^2 a_i + a_{i+1}^2 a_i^2 + a_{i+1}^4)(2A_i + a_{i+1}) - 2a_{i+1}^4 A_i - a_{i+1}^4 (a_{i+1} - a_i)] \\
= (a_{i+1} - a_i) [2a_{i+1}^4 a_i + a_{i+1}^3 a_i^2 + 2a_{i+1} a_i^3 A_i + 2a_{i+1} a_i^2 A_i + 2a_i^3 a_i A_i + a_{i+1} a_i A_i + a_{i+1} a_i^3] \\
> 0.
\]

We see that
\[
\sum_{n=1}^{\infty} a_n^3 \sum_{m=1}^{n} a_m A_m - \sum_{n=1}^{\infty} a_n^3 \sum_{m=1}^{n} a_m^2 A_m < \sum_{n=1}^{\infty} a_n^4 A_n^2 - \sum_{n=1}^{\infty} a_n^4 A_n^2.
\]

Thus, after each steps rearranges the non-negative sequence \(a\) to give the decreasing sequence, the left hand side of (4.1.1) is less some times than right hand side of (4.1.1). Therefore, the inequality (4.1.1) holds.

Now, we will find a constant \(K\) of inequality (4.1.1) with a decreasing sequence. We may assume that \(a_1 > 0\). Indeed, If \(a_1 = 0\), inequality (4.1.1) becomes
\[
\sum_{n=2}^{\infty} a_n^3 \sum_{m=1}^{n} a_m A_m \leq K \sum_{n=2}^{\infty} a_n^4 A_n^2.
\]
is weakly the inequality (4.1.1). The left hand side of (4.1.1) may be rewritten as
\[
L = \sum_{n=1}^{\infty} a_n^2 A_n \frac{a_n}{A_n} \sum_{m=1}^{n} a_m^2 A_m.
\]
Since the a’s is decreasing, we have
\[
a_n \frac{a_n}{A_n} \leq \frac{1}{n}.
\]
Thus,
\[
L \leq \sum_{n=1}^{\infty} a_n^2 A_n \frac{1}{n} \sum_{m=1}^{n} a_m^2 A_m.
\]
Applying the Holder’s inequality, we get
\[
\sum_{n=1}^{\infty} a_n^2 A_n \frac{1}{n} \sum_{m=1}^{n} a_m^2 A_m \leq \left( \sum_{n=1}^{\infty} a_n^4 A_n^2 \right)^{1/2} \left[ \sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{m=1}^{n} a_m^2 A_m \right)^2 \right]^{1/2}.
\]
Using the Hardy’s inequality, we also have
\[
\sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{m=1}^{n} a_m^2 A_m \right)^2 \leq 2 \sum_{n=1}^{\infty} a_n^4 A_n^2.
\]
Therefore
\[ L \leq 2 \sum_{n=1}^{\infty} a_n^4 A_n^2. \]

So that (4.1.1) holds with \( K = 2 \).

To handle the unrestricted version of (4.1.1) a substitute for Hardy’s inequality is called for, and this is what led to study weighted mean and factorable matrices. To illustrate these ideas we now prove a very general of version (4.1.1).

**Theorem 4.1.1.** Let \( p, q, r \geq 1 \). If \( (a_n)_{n=1}^{\infty} \) is a sequence of non-negative numbers with partial sum \( A_n = a_1 + \cdots + a_n \), then
\[
\sum_{n=1}^{\infty} a_n^p A_n^q \left( \sum_{m=n}^{\infty} a_m^{1+p/q} \right)^r \leq \left( \frac{p(q+r) - q}{p} \right)^r \sum_{n=1}^{\infty} (a_n^p A_n^q)^{1+r/q}.
\] (4.1.3)

**Proof.** We also assume that \( a_1 > 0 \) so \( A_n > 0 \). Let \( \lambda_n = a_n A_n^{q/p} ; y_n = a_n^{p/q} A_n \).

We have
\[
\Lambda_n = \lambda_1 + \cdots + \lambda_n \\
= a_1 A_1^{q/p} + \cdots + a_n A_n^{q/p} \\
\leq a_1 A_1^{q/p} + \cdots + a_n A_n^{q/p} \\
= A_n^{q/p+1}.
\]

Then
\[
1 \leq \frac{A_n^{q+1/p}}{\Lambda_n}.
\]

Therefore,
\[
\lambda_m^{1+p/q} \leq \frac{\lambda_m^{1+q/p} A_m^{q+1/q/p}}{\Lambda_m} = \frac{\lambda_m y_m}{\Lambda}.
\] (4.1.4)

Let
\[
\theta = \frac{r}{p(q+r) - q}.
\]

It is easy to see that \( 0 \leq \theta < 1 \). Using (4.1.4), we estimate left hand side of (4.1.3) so we obtain
\[
LHS \leq \sum_{n=1}^{\infty} \lambda_n^p \left( \sum_{m=n}^{\infty} \frac{\lambda_m y_m}{\Lambda_m} \right)^r.
\]

Applying the Holder’s inequality, we have
\[
\sum_{n=1}^{\infty} \lambda_n^{p(1-\theta)} \lambda_n^{p \theta} \left( \sum_{m=n}^{\infty} \frac{\lambda_m y_m}{\Lambda_m} \right)^r \leq \left( \sum_{n=1}^{\infty} \lambda_n^{p(1-\theta)/(1-p\theta)} \right)^{1-p\theta} \left[ \sum_{n=1}^{\infty} \lambda_n \left( \sum_{m=n}^{\infty} \frac{\lambda_m y_m}{\Lambda_m} \right)^{r/p\theta} \right]^{p\theta}.
\] (4.1.5)

We see that
\[
1 - \theta = \frac{p(q+r) - q - r}{p(q+r) - q - pr} = \frac{(p-1)(q+r)}{q(p-1)} = 1 + \frac{r}{p}.
\]
So that
\[ \lambda_n^{(1-\theta)/(1-p\theta)} = (a_n^p A_n^q)^{1+r/p} \] \hspace{1cm} (4.1.6)

Using Corollary 2.2.4 with \( c = \theta, p = r/(p\theta), x_m = y_m/\Lambda_m \) we have

\[
\left[ \sum_{n=1}^{\infty} \lambda_n \left( \sum_{m=n}^{\infty} \frac{\lambda_m y_m}{\Lambda_m} \right)^{r/(p\theta)} \right]^{p\theta} \leq \left[ \left( \frac{r}{(p\theta)} \right) \sum_{n=1}^{\infty} \lambda_n A_n^{r/(p\theta)} \left( \frac{y_n}{\Lambda_n} \right)^{r/(p\theta)} \right]^{p\theta} \\
= \left[ \frac{r}{p\theta} \left( \sum_{n=1}^{\infty} \lambda_n y_n^{r/(p\theta)} \right)^{p\theta} \right] \left( \sum_{n=1}^{\infty} \frac{a_n A_n^{q/p}}{a_n^{p/q} A_n^{q+r-q/p}} \right)^{p\theta} \\
= \left[ \frac{p(q+r) - q}{p} \right]^{r} \left( \sum_{n=1}^{\infty} a_n^{p(q+r)/q} A_n^{q+r} \right)^{p\theta}.
\]

Thus,
\[
\left[ \sum_{n=1}^{\infty} \lambda_n \left( \sum_{m=n}^{\infty} \frac{\lambda_m y_m}{\Lambda_m} \right)^{r/(p\theta)} \right]^{p\theta} \leq \left[ \frac{p(q+r) - q}{p} \right]^{r} \left( \sum_{n=1}^{\infty} \left( a_n^{p A_n^q} \right)^{1+r/q} \right)^{p\theta}.
\] \hspace{1cm} (4.1.7)

From (4.1.5), (4.1.6) and (4.1.7), we obtain

\[ \text{LHS} \leq \left[ \frac{p(q+r) - q}{p} \right]^{r} \sum_{n=1}^{\infty} \left( a_n^{p A_n^q} \right)^{1+r/q}. \]

We complete Theorem (4.1.1).

Inequalities [4.1.1] and [4.1.2] are also special cases. For \( p = 2, q = 1, r = 1 \) and by interchanging the order of summation on the left we get the inequality [4.1.1] and for \( q = r = 2, p = 1 \) we have the inequality [4.1.2].

4.2 Reverse Littlewood’s inequality

Does there exist a constant \( K \) such that
\[ \sum_{n=1}^{\infty} a_n^4 A_n^2 \leq K \sum_{n=1}^{\infty} a_n^3 \sum_{m=1}^{n} a_n^2 A_m ? \] \hspace{1cm} (4.2.1)

Unfortunately, the inequality (4.2.1) is false. This can be seen by taking
\[
a_n = \begin{cases} 
0 & \text{when } n > 2^N, \\
1 & \text{when } n = 2^r \text{ for } r = 1, \ldots, N, \\
e & \text{otherwise}.
\end{cases}
\] \hspace{1cm} (4.2.2)
For $2^r \leq n < 2^{r+1}$ so that

$$A_n = \sum_{s=1}^{r} a_{2s} + \sum_{\substack{1 \leq m < n \\ m \neq 2^s}} a_m = r + (n - r)\epsilon.$$  

We compute

$$LHS = \sum_{r=1}^{N} a_{2^r}^4 A_{2^r}^2 + \sum_{\substack{1 \leq n < 2N \\ n \neq 2^r}} a_n^4 A_n^2$$

$$> \sum_{r=1}^{N} [r + (2^r - r)\epsilon]^2$$

$$> \epsilon^2 \sum_{r=1}^{N} 2^{2r}$$

$$> \epsilon^2 2^{2N} = N^4$$  with taking $\epsilon = \frac{N^2}{2^N}$.

We also have

$$RHS = \sum_{n=1}^{\infty} a_n^3 \sum_{m=1}^{n} a_m^2 A_m$$

$$= \sum_{r=1}^{N} a_{2^r}^3 \sum_{m=1}^{2^r} a_m^2 A_m + \sum_{\substack{1 \leq n < 2N \\ n \neq 2^r}} a_n^3 \sum_{m=1}^{n} a_m^2 A_m$$

$$= \sum_{r=1}^{N} \sum_{m=1}^{2^r} a_m^2 A_m + \epsilon^3 \sum_{\substack{1 \leq n < 2N \\ n \neq 2^r}} \sum_{m=1}^{n} a_m^2 A_m$$

$$< \sum_{r=1}^{N} \sum_{s=1}^{r} a_{2^s}^2 A_{2^s} + \sum_{r=1}^{N} \sum_{\substack{1 \leq n < 2^r \\ n \neq 2^s}} a_n^2 A_n^2 + \epsilon^3 2^{2N} \sum_{m=1}^{2^N} a_m^2 A_m$$

$$< \sum_{r=1}^{N} \sum_{s=1}^{r} [s + (2^s - s)\epsilon] + \epsilon^2 \sum_{r=1}^{N} 2^r A_{2^r} + \epsilon^3 2^{2N} [N + 2^N \epsilon]$$

$$< (1 - \epsilon) \sum_{r=1}^{N} \frac{r(r + 1)}{2} + \epsilon \sum_{r=1}^{N} (2^{r+1} - 2) + \epsilon^2 \sum_{r=1}^{N} 2^r (r + 2^r \epsilon) + \epsilon^3 2^{2N} [N + 2^N \epsilon]$$

$$< N^3 + \epsilon N 2^{N+1} + \epsilon^2 N 2^N (N + 2^N \epsilon) + \epsilon^3 2^{2N} (N + 2^N \epsilon)$$

$$< 6N^3$$  with $\epsilon = \frac{N^2}{2^N}$.

Thus, $LHS > RHS$ with choosing above $a_n$. We have pointed out there dose not exist $K$ inequality (4.2.2). Now, we consider for general case

$$\sum_{n=1}^{\infty} (a_n^p A_n^q)^{1+r/p} \leq K \sum_{n=1}^{\infty} a_n^p (\sum_{m=n}^{\infty} a_m^{1+p/q})^r. \quad (4.2.3)$$
We also take
\[ a_n = \begin{cases} 
0 & \text{if } n > 2^N, \\
1 & \text{if } n = 2^r \text{ for } r = 1, 2, \ldots, N, \\
\epsilon & \text{otherwise}.
\end{cases} \]

We estimate the left hand side
\[
LHS = \sum_{n=1}^{2^N} (a\hat{n}^p A_n^q)^{1+r/q}
\]
\[
= \sum_{s=1}^{N} a_{2^s A_{2^s}}^p A_{2^s}^q + \sum_{1 \leq n < 2^N \atop n \neq 2^s} (a^n A_n^q)^{1+r/q}
\]
\[
> \sum_{s=1}^{N} [s + (2^s - s)\epsilon]^{q+r}
\]
\[
= \sum_{s=1}^{N} [(1 - \epsilon)^{q+r} s^{q+r} + 2^{s(q+r)} \epsilon^{q+r}]
\]
\[
= (1 - \epsilon)^{q+r} \sum_{s=1}^{N} s^{q+r} + \epsilon^{q+r} \sum_{s=1}^{N} 2^{s(q+r)}
\]
\[
= (1 - \epsilon)^{q+r} N^{q+r} + \epsilon^{q+r} 2^{N(q+r)}
\]
\[
= N^{q+r} + N^{\alpha(q+r)} \text{ with } \epsilon = \frac{N^\alpha}{2^N}.
\]

We also have
\[
RHS = \sum_{n=1}^{2^N} a\hat{n}^p A_n^q \left( \sum_{m=n}^{2^N} a_m^{1+p/q} \right)^r
\]
\[
= \sum_{s=1}^{N} a_{2^s A_{2^s}}^p A_{2^s}^q \left( \sum_{m=2^s}^{2^N} a_m^{1+p/q} \right)^r + \sum_{1 \leq n < 2^N \atop n \neq 2^s} a^n A_n^q \left( \sum_{m=n}^{2^N} a_m^{1+p/q} \right)^r
\]
\[
= I_1 + \epsilon^p I_2.
\]
We continue to compute $I_1$.\[I_1 = \sum_{s=1}^{N} [s + (2^s - s)\epsilon]^q [N + 1 + (2^N - 2^s)\epsilon^{1+p/q}]^r < \sum_{s=1}^{N} (s + 2^s\epsilon)^q \left( N + 1 + 2^N \epsilon^{1+p/q} \right)^r < \sum_{s=1}^{N} 2^q (s^q + 2^{sq}\epsilon^q) 2^r \left( (N + 1)^r + 2^{Nq} \epsilon^{r(1+p/q)} \right) < 2^{q+r} \left( (2N)^r + 2^{Nq} \epsilon^{r(1+p/q)} \right) \sum_{s=1}^{N} (s^q + 2^{sq}\epsilon^q) = 2^{q+r} \left( (2N)^r + 2^{Nq} \epsilon^{r(1+p/q)} \right) N \left( \sum_{s=1}^{N} N^q + \epsilon^q \sum_{s=1}^{N} 2^{sq} \right) < 2^{q+r} \left( (2N)^r + 2^{Nq} \epsilon^{r(1+p/q)} \right) N \left( N^q + \epsilon^q \frac{2^{(N+1)q}}{2^q - 1} - \frac{2^q}{2^q - 1} \right) < 2^{q+r} \left( (2N)^r + 2^{Nq} \epsilon^{r(1+p/q)} \right) N \left( N^q + \epsilon^q \frac{2^{(N+1)q}}{2^q - 1} \right) = 2^{q+r} N \left( 2^r N^r + 2^{Nq} \left( \frac{N^q}{2^N} \right)^{r(1+p/q)} \right) \left[ N^q + \epsilon^q \frac{2^q N^{aq}}{2^q - 1} \right] with \ \epsilon = \frac{N^a}{2^N} < 2^{q+r} N \left( 2^r N^r + 2^{Nq} \left( \frac{N^q}{2^N} \right)^{r(1+p/q)} \right) \left[ \frac{2^q N^{aq}}{2^q - 1} \right] N^q \left( 2^r N^r + 2^{Nq} \left( \frac{N^q}{2^N} \right)^{r(1+p/q)} \right) \left[ \frac{2^q N^{aq}}{2^q - 1} \right] < \frac{2^{q+2r+2}}{2^q - 1} N^{aq+r}.

We also have\[I_2 = \sum_{1 \leq n < 2^N \atop n \neq 2^s} A_n^q \left( \sum_{m=n}^{2^N} a_{m}^{1+p/q} \right)^r = \sum_{s=0}^{N-1} 2^s A_{2^s+1}^q \left( \sum_{m=1}^{2^N} a_{m}^{1+p/q} \right)^r < \sum_{s=0}^{N-1} 2^s \left( s + 1 + 2^{s+1} \epsilon \right)^q \left[ N + 2^N \epsilon^{1+p/q} \right]^r < \sum_{s=0}^{N-1} 2^s 2^q \left( (s + 1)^q + 2^{q(s+1)} \epsilon^q \right) \left[ N^r + 2^{Nq} \epsilon^{r(1+p/q)} \right] .\]
\[ I_2 < 2^{q+r} \left[ N^r + 2^N r^{(1+p/q)} \right] \left[ \sum_{s=0}^{N-1} 2^s (s + 1)^q + \epsilon \sum_{s=0}^{N-1} 2^s 2^q(s+1) \right] \]
\[ \leq 2^{q+r} \left[ N^r + 2^N r^{(1+p/q)} \right] \left( 2^N N^q + \epsilon 2^N N^q(q+1) \right) \]
\[ < 2^{q+r} \left[ N^r + 2^N r \left( \frac{N^\alpha}{2^N} \right)^{(1+p/q)} \right] \left( 2^N N^q + 2^N N^{\alpha q} \right) \quad \text{with} \quad \epsilon = \frac{N^\alpha}{2^N} \]
\[ < 2^{q+r+1} 2^N N^{\alpha q + r}. \]

If \( p > 1, r > 1 \), then
\[ \epsilon^p I_2 = 2^{q+r+1} N^{\alpha p + \alpha q + r} \frac{2N(p-1)}{2^N} \to 0 \quad \text{as} \quad N \to \infty. \]

Therefore,
\[ \text{RHS} < \frac{2^{q+2r+2}}{2^q - 1} N^{\alpha q + r}. \]

Thus, we need to choose \( \alpha > 1 \).

If \( p > 1, r = 1 \), then
\[ \text{LHS} > N^{q+1} + N^{\alpha(q+1)}, \]
\[ \text{RHS} < \frac{2^{q+1}}{2^q - 1} N^{\alpha(q+1)}. \]

Hence, we need to choose \( \alpha > 1 \).

If \( p = 1, r > 1 \), then
\[ \epsilon^p I_2 = 2^{q+r+1} N^{\alpha + \alpha q + r}. \]

And
\[ \text{LHS} > N^{q+r} + N^{\alpha(q+r)}, \]
\[ \text{RHS} < \frac{2^{q+2r+2}}{2^q - 1} N^{\alpha q + r} + 2^{q+r+1} N^{\alpha + \alpha q + r}. \]

Thus, we choose
\[ \alpha > \frac{r}{r - 1}. \]

If \( r = 1, p = 1 \), then
\[ \text{LHS} = \sum_{n=1}^{\infty} (a_n A_n^q)^{1+1/q} = \sum_{m=1}^{\infty} a_m^{1+1/q} A_m^{q+1}. \]

And
\[ \text{RHS} = \sum_{n=1}^{\infty} a_n A_n^q \sum_{m=n}^{\infty} a_m^{1+1/q} = \sum_{m=1}^{\infty} a_m^{1+1/q} \sum_{n=1}^{m} a_n A_n^q. \]

So we need only to compare \( A_m^{q+1} \) and \( \sum_{n=1}^{m} a_n A_n^q \). By Power Rule Lemma, we always have
\[ \sum_{n=1}^{m} a_n A_n^q \leq A_m^{q+1} \leq q \sum_{n=1}^{m} a_n A_n^q. \]

Thus we always find a constant \( K \) in inequality \([4.2.3]\) with \( q = 1, r = 1 \). Summary, we have not found \( K \) to the Littlewood’s inequality reserve direction in general case. And in the last part, we will give the reverse Littlewood’s inequality with a decreasing sequence.
CHAPTER 4. LITTLEWOOD’S PROBLEM

Lemma 4.2.1. Let $s, t \geq 0$. If $a$ is an $N$–tuple of non-negative number with partial sums $A_n = a_1 + \cdots + a_n$, then

$$\sum_{n=1}^{N} a_n A_n^s \left( \sum_{m=n}^{N} a_m \right)^t \geq \beta(s + 1, t + 1) A_{N}^{s+t+1}$$

(4.2.4)

where

$$\beta(s + 1, t + 1) = \int_{0}^{1} y^s (1 - y)^t dy.$$ 

Proof. Let $A_0 = 0$. We have $A_{n-1} \leq x \leq A_n$ so that

$$A_N - x \leq A_N - A_{n-1} = \sum_{m=n}^{N} a_m.$$ 

Thus,

$$\int_{A_{n-1}}^{A_n} x^s (A_N - x)^t dx \leq A_n^s \left( \sum_{m=n}^{N} a_m \right)^t \int_{A_{n-1}}^{A_n} dx = a_n A_n^s \left( \sum_{m=n}^{N} a_m \right)^t .$$

Therefore,

$$\sum_{n=1}^{N} a_n A_n^s \left( \sum_{m=n}^{N} a_m \right)^t \geq \int_{0}^{A_n} x^s (A_N - x)^t dx$$

$$= \int_{0}^{1} (yA_N)^s (A_N - yA_n)^t d(yA_N)$$

$$= A_{N}^{s+t+1} \int_{0}^{1} y^s (1 - y)^t dy.$$ 

We are done. 

Theorem 4.2.1. Let $p, q, r \geq 1$. If $(a_n)_{n=1}^{\infty}$ is a decreasing sequence of non-negative number with partial sum $A_n = a_1 + \cdots + a_n$, then

$$\sum_{n=1}^{\infty} (a_n A_n^q)^{1+r/q} \leq \frac{1}{\beta(q + 1, r)} \sum_{n=1}^{\infty} a_n A_n^{q} \left( \sum_{m=n}^{\infty} a_m^{1+p/q} \right)^r .$$

(4.2.5)

Proof. Applying Lemma 4.2.1 with $s = q, t = r - 1$ we obtain for $m = 1, 2, ..$

$$\beta(q + 1, r) A_m^{q+r} \leq \sum_{n=1}^{m} a_n A_n^{q} \left( \sum_{j=n}^{m} a_j \right)^{r-1} .$$

So that

$$\beta(q + 1, r) a_m^{p+pr/q} A_m^{q+r} \leq a_m^{p+pr/q} \sum_{n=1}^{m} a_n A_n^{q} \left( \sum_{j=n}^{m} a_j \right)^{r-1} .$$
Since \( a \)’s are decreasing sequence, then
\[
a_m^{p+pr/q} \sum_{n=1}^{m} a_n A_n^q \left( \sum_{j=n}^{m} a_j \right)^{r-1} = a_m^{1+p/q} a_m^{p-1} \left( \sum_{n=1}^{m} a_n A_n^q \right) \left( \sum_{j=n}^{m} a_j \right)^{r-1}
\]
\[
\leq a_m^{1+p/q} a_m^{p-1} \left( \sum_{n=1}^{m} a_n A_n^q \right) \left( \sum_{j=n}^{m} a_j \right)^{r-1}
\]
\[
= a_m^{1+p/q} \sum_{n=1}^{m} a_n^P A_n^q \left( \sum_{j=n}^{m} a_j^{1+p/q} \right)^{r-1}
\]
Therefore,
\[\beta(q+1,r) a_m^{p+pr/q} A_m^q + r \leq a_m^{1+p/q} \sum_{n=1}^{m} a_n^P A_n^q \left( \sum_{j=n}^{m} a_j^{1+p/q} \right)^{r-1}\]

Summing on \( m \), we get
\[
\beta(q+1,r) \sum_{m=1}^{\infty} (a_m^P A_m^q)^{1+r/q} \leq \sum_{m=1}^{\infty} a_m^{1+p/q} \sum_{n=1}^{m} a_n^P A_n^q \left( \sum_{j=n}^{m} a_j^{1+p/q} \right)^{r-1}
\]
\[
= \sum_{n=1}^{\infty} a_n^P A_n^q \sum_{m=n}^{\infty} a_m^{1+p/q} \left( \sum_{j=n}^{m} a_j^{1+p/q} \right)^{r-1}
\]
\[
\leq \sum_{n=1}^{\infty} a_n^P A_n^q \left( \sum_{m=n}^{\infty} a_m^{1+p/q} \right)^r
\]
We are done.
Conclusion

In this thesis, we have shown that extensive Hardy’s inequality II in case $r > s \geq 1$, but in other cases $0 < r \leq s$ and $0 < s < 1, r > s$ it is not true. In Chapter 1, we have not known the best constants of two Hardy’s extensive inequalities.

In Chapter 2 and 3, we have applied extensive Harydy’s inequality II to give some conditions such that the factorable matrix and weighted mean matrix are bounded operators from $l^p$ into $l^q$. In addition, many Corollaries are involved the factorable matrices and the weighted mean matrices. Besides, we have given an special factor matrix in section 2.3 and have found the conditions to it is bounded operator. Moreover, we have pointed out the norm of it for $p = 1, q = \infty$, but we have not computed the norm of this factorable matrix case $p > 1, q = \infty$ or $1 < p \leq q < \infty$ or $p > q$. In case $p = 1$ and $1 \leq q < \infty$, we do not find the condition of $\alpha$ and $\beta$ to the factorable matrix in section 2.3 is bounded operator. Next, for the weighted mean matrix, we have determined the norm of it in case $p = 1, q = \infty$ and in case $p > 1, q = \infty$. In other cases, we have not computed the norm of the weighted mean matrix. In last section of Chapter 3, we gave some counterexamples to point out the inverse of some Corollaries does not hold.

Finally, in Chapter 4 we consider the Littlewood’s problem and found a constant $K$ but it is not the best constant of Littlewood’s problem. We have proved the general case of Littlewood’s inequality and we give account example for reverse Littlewood’s inequality. We also have shown that the reverse Littewood’s inequality for the decreasing sequence. Can we add other conditions to the reverse Littlewood’s inequality?
Bibliography


