13.7 Exercises

1. A hemisphere $H$ and a portion $P$ of a paraboloid are shown. Suppose $F$ is a vector field on $\mathbb{R}^3$ whose components have continuous partial derivatives. Explain why

$$\int\int_S \text{curl}\, F \cdot dS = \int\int_C F \cdot d\mathbf{r}$$

2–6 Use Stokes’ Theorem to evaluate $\int\int_S \text{curl}\, F \cdot dS$.

2. $F(x, y, z) = yz \mathbf{i} + xz \mathbf{j} + xy \mathbf{k}$,
   $S$ is the part of the paraboloid $z = 9 - x^2 - y^2$ that lies above the plane $z = 5$, oriented upward

3. $F(x, y, z) = x^2 e^{-y} \mathbf{i} + y^2 e^{-x} \mathbf{j} + z^2 e^{-x} \mathbf{k}$,
   $S$ is the hemisphere $x^2 + y^2 + z^2 = 4$, $z \geq 0$, oriented upward

4. $F(x, y, z) = (x + \tan^{-1} yz) \mathbf{i} + yz \mathbf{j} + z \mathbf{k}$,
   $S$ is the part of the hemisphere $x = \sqrt{9 - y^2 - z^2}$ that lies inside the cylinder $y^2 + z^2 = 4$, oriented in the direction of the positive $x$-axis

5. $F(x, y, z) = xyz \mathbf{i} + xy \mathbf{j} + x^2 yz \mathbf{k}$,
   $S$ consists of the top and the four sides (but not the bottom) of the cube with vertices $(\pm 1, \pm 1, \pm 1)$, oriented outward [Hint: Use Exercise 3.]

6. $F(x, y, z) = xy \mathbf{i} + e^z \mathbf{j} + xyz \mathbf{k}$,
   $S$ consists of the four sides of the pyramid with vertices $(0, 0, 0), (1, 0, 0), (0, 0, 1), (1, 0, 1)$, and $(0, 1, 0)$ that lie to the right of the $xz$-plane, oriented in the direction of the positive $y$-axis [Hint: Use Exercise 3.]

7–10 Use Stokes’ Theorem to evaluate $\int_C F \cdot d\mathbf{r}$. In each case $C$ is oriented counterclockwise as viewed from above.

7. $F(x, y, z) = (x + y^2) \mathbf{i} + (y + z^2) \mathbf{j} + (z + x^2) \mathbf{k}$,
   $C$ is the triangle with vertices $(1, 0, 0), (0, 1, 0)$, and $(0, 0, 1)$

8. $F(x, y, z) = e^{-x} \mathbf{i} + e^{-y} \mathbf{j} + e^{-z} \mathbf{k}$,
   $C$ is the boundary of the part of the plane $2x + y + 2z = 2$ in the first octant

9. $F(x, y, z) = 2z \mathbf{i} + 4x \mathbf{j} + 5y \mathbf{k}$,
   $C$ is the curve of intersection of the plane $z = x + 4$ and the cylinder $x^2 + y^2 = 4$

10. $F(x, y, z) = x \mathbf{i} + y \mathbf{j} + (x^2 + y^2) \mathbf{k}$,
    $C$ is the boundary of the part of the paraboloid $z = 1 - x^2 - y^2$ in the first octant

11. (a) Use Stokes’ Theorem to evaluate $\int_C F \cdot d\mathbf{r}$, where
    $$F(x, y, z) = x^2 z \mathbf{i} + xy^2 \mathbf{j} + z^2 \mathbf{k}$$
    and $C$ is the curve of intersection of the plane $x + y + z = 1$ and the cylinder $x^2 + y^2 = 9$ oriented counterclockwise as viewed from above.
    (b) Graph both the plane and the cylinder with domains chosen so that you can see the curve $C$ and the surface that you used in part (a).
    (c) Find parametric equations for $C$ and use them to graph $C$.

12. (a) Use Stokes’ Theorem to evaluate $\int_C F \cdot d\mathbf{r}$, where
    $$F(x, y, z) = x^2 y \mathbf{i} + \frac{1}{2} x^2 \mathbf{j} + xy \mathbf{k}$$
    and $C$ is the curve of intersection of the hyperbolic paraboloid $z = y^2 - x^2$ and the cylinder $x^2 + y^2 = 1$ oriented counterclockwise as viewed from above.
    (b) Graph both the hyperbolic paraboloid and the cylinder with domains chosen so that you can see the curve $C$ and the surface that you used in part (a).
    (c) Find parametric equations for $C$ and use them to graph $C$.

13–15 Verify that Stokes’ Theorem is true for the given vector field $F$ and surface $S$.

13. $F(x, y, z) = y^3 \mathbf{i} + x \mathbf{j} + z^2 \mathbf{k}$,
   $S$ is the part of the paraboloid $z = 1 - x^2 + y^2$ that lies below the plane $z = 1$, oriented upward

14. $F(x, y, z) = x \mathbf{i} + y \mathbf{j} + xy \mathbf{k}$,
   $S$ is the part of the plane $2x + y + z = 2$ that lies in the first octant, oriented upward

15. $F(x, y, z) = y \mathbf{i} + z \mathbf{j} + x \mathbf{k}$,
   $S$ is the hemisphere $x^2 + y^2 + z^2 = 1$, $z \geq 0$, oriented in the direction of the positive $y$-axis

16. Let
    $$F(x, y, z) = (ax^3 - 3x^2z, x^3y + by^3, cz^3)$$
    Let $C$ be the curve in Exercise 12 and consider all possible smooth surfaces $S$ whose boundary curve is $C$. Find the values of $a$, $b$, and $c$ for which $\int_C F \cdot dS$ is independent of the choice of $S$. 

17. Calculate the work done by the force field
\[ \mathbf{F}(x, y, z) = (x^2 + z^2) \mathbf{i} + (y^2 + x^2) \mathbf{j} + (z^2 + y^2) \mathbf{k} \]
when a particle moves under its influence around the edge of the part of the sphere \( x^2 + y^2 + z^2 = 4 \) that lies in the first octant, in a counterclockwise direction as viewed from above.

18. Evaluate \( \int_C (y + \sin x) \, dx + (z^2 + \cos y) \, dy + x^3 \, dz \)
where \( C \) is the curve \( r(t) = (\sin t, \cos t, \sin 2t) \), \( 0 \leq t \leq 2\pi \). \[ \text{[Hint: Observe that } \mathbf{C} \text{ lies on the surface } z = 2xy.\]}

19. If \( S \) is a sphere and \( \mathbf{F} \) satisfies the hypotheses of Stokes’ Theorem, show that \( \int_S \text{curl } \mathbf{F} \cdot dS = 0 \).

20. Suppose \( S \) and \( C \) satisfy the hypotheses of Stokes’ Theorem and \( f, g \) have continuous second-order partial derivatives. Use Exercises 22 and 24 in Section 13.5 to show the following.
(a) \( \int_C (f \nabla g) \cdot d\mathbf{r} = \int_S (\nabla f \times \nabla g) \cdot dS \)
(b) \( \int_C (f \nabla \mathbf{f}) \cdot d\mathbf{r} = 0 \)
(c) \( \int_C (f \nabla g + g \nabla f) \cdot d\mathbf{r} = 0 \)

Three Men and Two Theorems

Although two of the most important theorems in vector calculus are named after George Green and George Stokes, a third man, William Thomson (also known as Lord Kelvin), played a large role in the formulation, dissemination, and application of both of these results. All three men were interested in how the two theorems could help to explain and predict physical phenomena in electricity and magnetism and fluid flow. The basic facts of the story are given in the margin notes on pages 946 and 972.

Write a report on the historical origins of Green’s Theorem and Stokes’ Theorem. Explain the similarities and relationship between the theorems. Discuss the roles that Green, Thomson, and Stokes played in discovering these theorems and making them widely known. Show how both theorems arose from the investigation of electricity and magnetism and were later used to study a variety of physical problems.


3. See the article on Green by P. J. Wallis in Volume XV and the articles on Thomson by Jed Buchwald and on Stokes by E. M. Parkinson in Volume XIII.