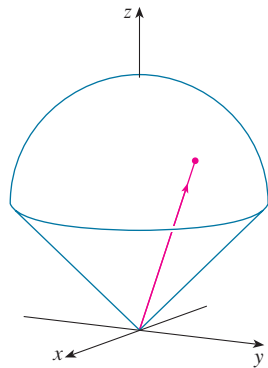
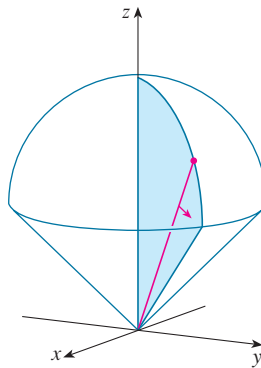


Figure 11 shows how E is swept out if we integrate first with respect to ρ , then ϕ , and then θ . The volume of E is

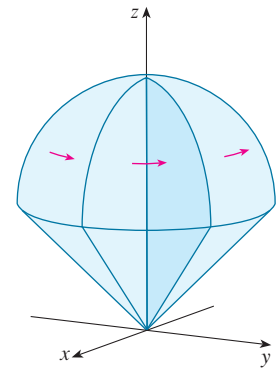
$$\begin{aligned} V(E) &= \iiint_E dV = \int_0^{2\pi} \int_0^{\pi/4} \int_0^{\cos \phi} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \\ &= \int_0^{2\pi} d\theta \int_0^{\pi/4} \sin \phi \left[\frac{\rho^3}{3} \right]_{\rho=0}^{\rho=\cos \phi} d\phi \\ &= \frac{2\pi}{3} \int_0^{\pi/4} \sin \phi \cos^3 \phi \, d\phi = \frac{2\pi}{3} \left[-\frac{\cos^4 \phi}{4} \right]_0^{\pi/4} = \frac{\pi}{8} \end{aligned}$$



ρ varies from 0 to $\cos \phi$ while ϕ and θ are constant.



ϕ varies from 0 to $\pi/4$ while θ is constant.



θ varies from 0 to 2π .

FIGURE 11

12.8

Exercises

1–4 ■ Sketch the solid whose volume is given by the integral and evaluate the integral.

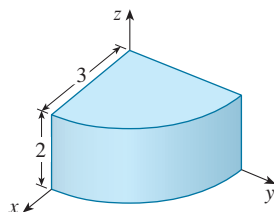
1. $\int_0^4 \int_0^{2\pi} \int_r^4 r \, dz \, d\theta \, dr$ 2. $\int_0^{\pi/2} \int_0^2 \int_0^{9-r^2} r \, dz \, dr \, d\theta$

3. $\int_0^{\pi/6} \int_0^{\pi/2} \int_0^3 \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi$

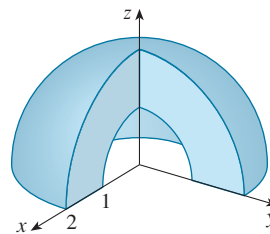
4. $\int_0^{2\pi} \int_{\pi/2}^{\pi} \int_1^2 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$

5–6 ■ Set up the triple integral of an arbitrary continuous function $f(x, y, z)$ in cylindrical or spherical coordinates over the solid shown.

5.



6.



7–14 ■ Use cylindrical coordinates.

7. Evaluate $\iiint_E \sqrt{x^2 + y^2} \, dV$, where E is the region that lies inside the cylinder $x^2 + y^2 = 16$ and between the planes $z = -5$ and $z = 4$.

8. Evaluate $\iiint_E (x^3 + xy^2) \, dV$, where E is the solid in the first octant that lies beneath the paraboloid $z = 1 - x^2 - y^2$.

9. Evaluate $\iiint_E y \, dV$, where E is the solid that lies between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$, above the xy -plane, and below the plane $z = x + 2$.

10. Evaluate $\iiint_E xz \, dV$, where E is bounded by the planes $z = 0$, $z = y$, and the cylinder $x^2 + y^2 = 1$ in the half-space $y \geq 0$.
11. Evaluate $\iiint_E x^2 \, dV$, where E is the solid that lies within the cylinder $x^2 + y^2 = 1$, above the plane $z = 0$, and below the cone $z^2 = 4x^2 + 4y^2$.
12. (a) Find the volume of the solid that the cylinder $r = a \cos \theta$ cuts out of the sphere of radius a centered at the origin.
 (b) Illustrate the solid of part (a) by graphing the sphere and the cylinder on the same screen.
13. Find the mass and center of mass of the solid S bounded by the paraboloid $z = 4x^2 + 4y^2$ and the plane $z = a$ ($a > 0$) if S has constant density K .
14. (a) Find the volume of the region E bounded by the paraboloids $z = x^2 + y^2$ and $z = 36 - 3x^2 - 3y^2$.
 (b) Find the centroid of the region E in part (a).

25–24 ■ Use spherical coordinates.

15. Evaluate $\iiint_B (x^2 + y^2 + z^2) \, dV$, where B is the unit ball $x^2 + y^2 + z^2 \leq 1$.
16. Evaluate $\iiint_H (x^2 + y^2) \, dV$, where H is the hemispherical region that lies above the xy -plane and below the sphere $x^2 + y^2 + z^2 = 1$.
17. Evaluate $\iiint_E z \, dV$, where E lies between the spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 4$ in the first octant.
18. Evaluate $\iiint_E x e^{(x^2+y^2+z^2)^2} \, dV$, where E is the solid that lies between the spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 4$ in the first octant.
19. Evaluate $\iiint_E \sqrt{x^2 + y^2 + z^2} \, dV$, where E is bounded below by the cone $\phi = \pi/6$ and above by the sphere $\rho = 2$.
20. Find the volume of the solid that lies within the sphere $x^2 + y^2 + z^2 = 4$, above the xy -plane, and below the cone $z = \sqrt{x^2 + y^2}$.
21. (a) Find the volume of the solid that lies above the cone $\phi = \pi/3$ and below the sphere $\rho = 4 \cos \phi$.
 (b) Find the centroid of the solid in part (a).
22. Let H be a solid hemisphere of radius a whose density at any point is proportional to its distance from the center of the base.
 (a) Find the mass of H .
 (b) Find the center of mass of H .
 (c) Find the moment of inertia of H about its axis.
23. (a) Find the centroid of a solid homogeneous hemisphere of radius a .
 (b) Find the moment of inertia of the solid in part (a) about a diameter of its base.

24. Find the mass and center of mass of a solid hemisphere of radius a if the density at any point is proportional to its distance from the base.

25–28 ■ Use cylindrical or spherical coordinates, whichever seems more appropriate.

25. Find the volume and centroid of the solid E that lies above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = 1$.
26. Find the volume of the smaller wedge cut from a sphere of radius a by two planes that intersect along a diameter at an angle of $\pi/6$.
27. Evaluate $\iiint_E z \, dV$, where E lies above the paraboloid $z = x^2 + y^2$ and below the plane $z = 2y$. Use either the Table of Integrals (on the back Reference Pages) or a computer algebra system to evaluate the integral.
28. (a) Find the volume enclosed by the torus $\rho = \sin \phi$.
 (b) Use a computer to draw the torus.

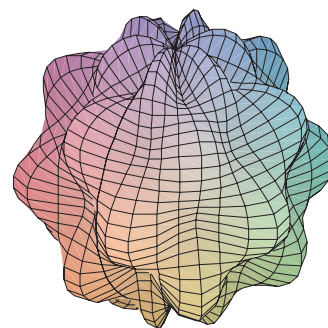
29. Evaluate the integral by changing to cylindrical coordinates:

$$\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{x^2+y^2}^{2-x^2-y^2} (x^2 + y^2)^{3/2} \, dz \, dy \, dx$$

30. Evaluate the integral by changing to spherical coordinates:

$$\int_0^3 \int_0^{\sqrt{9-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{18-x^2-y^2}} (x^2 + y^2 + z^2) \, dz \, dx \, dy$$

31. In the Laboratory Project on page 699 we investigated the family of surfaces $\rho = 1 + \frac{1}{5} \sin m\theta \sin n\phi$ that have been used as models for tumors. The “bumpy sphere” with $m = 6$ and $n = 5$ is shown. Use a computer algebra system to find its volume.



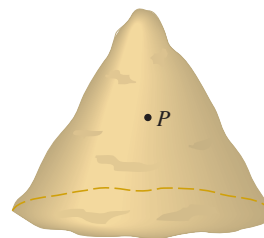
32. Show that

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sqrt{x^2 + y^2 + z^2} e^{-(x^2+y^2+z^2)} \, dx \, dy \, dz = 2\pi$$

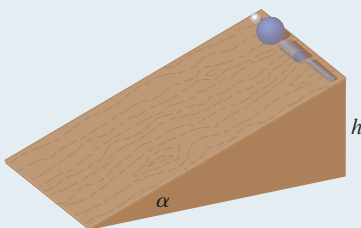
(The improper triple integral is defined as the limit of a triple integral over a solid sphere as the radius of the sphere increases indefinitely.)

33. When studying the formation of mountain ranges, geologists estimate the amount of work required to lift a mountain from sea level. Consider a mountain that is essentially in the shape of a right circular cone. Suppose that the weight density of the material in the vicinity of a point P is $g(P)$ and the height is $h(P)$.
- Find a definite integral that represents the total work done in forming the mountain.
 - Assume that Mount Fuji in Japan is in the shape of a right circular cone with radius 62,000 ft, height 12,400 ft, and density a constant 200 lb/ft^3 . How

much work was done in forming Mount Fuji if the land was initially at sea level?



Applied Project



Roller Derby

Suppose that a solid ball (a marble), a hollow ball (a squash ball), a solid cylinder (a steel bar), and a hollow cylinder (a lead pipe) roll down a slope. Which of these objects reaches the bottom first? (Make a guess before proceeding.)

To answer this question we consider a ball or cylinder with mass m , radius r , and moment of inertia I (about the axis of rotation). If the vertical drop is h , then the potential energy at the top is mgh . Suppose the object reaches the bottom with velocity v and angular velocity ω , so $v = \omega r$. The kinetic energy at the bottom consists of two parts: $\frac{1}{2}mv^2$ from translation (moving down the slope) and $\frac{1}{2}I\omega^2$ from rotation. If we assume that energy loss from rolling friction is negligible, then conservation of energy gives

$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

- Show that

$$v^2 = \frac{2gh}{1 + I^*} \quad \text{where } I^* = \frac{I}{mr^2}$$

- If $y(t)$ is the vertical distance traveled at time t , then the same reasoning as used in Problem 1 shows that $v^2 = 2gy/(1 + I^*)$ at any time t . Use this result to show that y satisfies the differential equation

$$\frac{dy}{dt} = \sqrt{\frac{2g}{1 + I^*}} (\sin \alpha) \sqrt{y}$$

where α is the angle of inclination of the plane.

- By solving the differential equation in Problem 2, show that the total travel time is

$$T = \sqrt{\frac{2h(1 + I^*)}{g \sin^2 \alpha}}$$

This shows that the object with the smallest value of I^* wins the race.

- Show that $I^* = \frac{1}{2}$ for a solid cylinder and $I^* = 1$ for a hollow cylinder.
- Calculate I^* for a partly hollow ball with inner radius a and outer radius r . Express your answer in terms of $b = a/r$. What happens as $a \rightarrow 0$ and as $a \rightarrow r$?
- Show that $I^* = \frac{2}{5}$ for a solid ball and $I^* = \frac{2}{3}$ for a hollow ball. Thus, the objects finish in the following order: solid ball, solid cylinder, hollow ball, hollow cylinder.