▲ Instead of using tables, we could have used the identity $\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$ twice.

Now we use Formula 64 in the Table of Integrals:

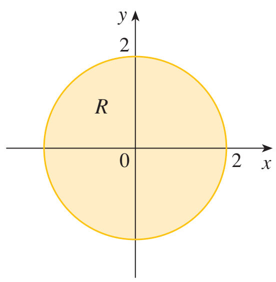
$$V = 6\int_0^{\pi/2} \cos^2\theta \, d\theta = 6\left[\tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta\right]_0^{\pi/2}$$

$$= 6 \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{3\pi}{2}$$

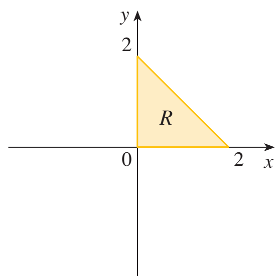
12.4 Exercises

1–6 ■ A region R is shown. Decide whether to use polar coordinates or rectangular coordinates and write $\iint_R f(x, y)\, dA$ as an iterated integral, where f is an arbitrary continuous function on R.

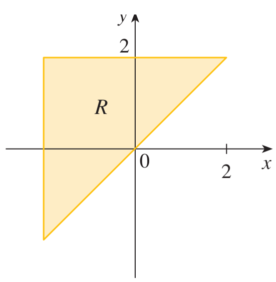
1.



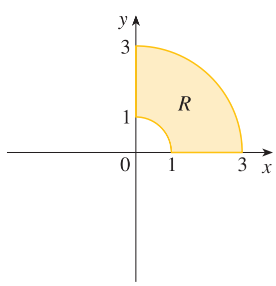
2.



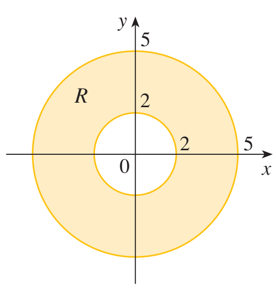
3.



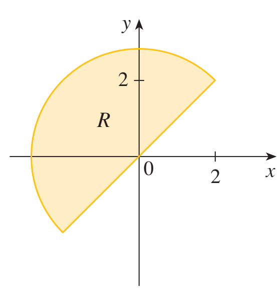
4.



5.



6.



7–8 ■ Sketch the region whose area is given by the integral and evaluate the integral.

7. $\int_{\pi}^{2\pi}\int_4^7 r\, dr\, d\theta$

8. $\int_0^{\pi/2}\int_0^{4\cos\theta} r\, dr\, d\theta$

9–14 ■ Evaluate the given integral by changing to polar coordinates.

9. $\iint_D xy\, dA$,
where D is the disk with center the origin and radius 3

10. $\iint_R \sqrt{x^2 + y^2}\, dA$,
where $R = \{(x, y) \mid 1 \leqslant x^2 + y^2 \leqslant 9,\ y \geqslant 0\}$

11. $\iint_D e^{-x^2-y^2}\, dA$, where D is the region bounded by the semicircle $x = \sqrt{4 - y^2}$ and the y-axis

12. $\iint_R ye^x\, dA$, where R is the region in the first quadrant enclosed by the circle $x^2 + y^2 = 25$

13. $\iint_R \arctan(y/x)\, dA$,
where $R = \{(x, y) \mid 1 \leqslant x^2 + y^2 \leqslant 4,\ -x \leqslant y \leqslant x\}$

14. $\iint_D x\, dA$, where D is the region in the first quadrant that lies between the circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 2x$

15–21 ■ Use polar coordinates to find the volume of the given solid.

15. Under the paraboloid $z = x^2 + y^2$ and above the disk $x^2 + y^2 \leqslant 9$

16. Inside the sphere $x^2 + y^2 + z^2 = 16$ and outside the cylinder $x^2 + y^2 = 4$

17. A sphere of radius a

18. Bounded by the paraboloid $z = 10 - 3x^2 - 3y^2$ and the plane $z = 4$

19. Above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = 1$

20. Bounded by the paraboloids $z = 3x^2 + 3y^2$ and $z = 4 - x^2 - y^2$

21. Inside both the cylinder $x^2 + y^2 = 4$ and the ellipsoid $4x^2 + 4y^2 + z^2 = 64$

22. (a) A cylindrical drill with radius r_1 is used to bore a hole through the center of a sphere of radius r_2. Find the volume of the ring-shaped solid that remains.
(b) Express the volume in part (a) in terms of the height h of the ring. Notice that the volume depends only on h, not on r_1 or r_2.

23–24 ■ Use a double integral to find the area of the region.

23. One loop of the rose $r = \cos 3\theta$

24. The region enclosed by the cardioid $r = 1 - \sin\theta$

25–28 ■ Evaluate the iterated integral by converting to polar coordinates.

25. $\int_0^1 \int_0^{\sqrt{1-x^2}} e^{x^2+y^2}\,dy\,dx$

26. $\int_{-a}^a \int_0^{\sqrt{a^2-y^2}} (x^2+y^2)^{3/2}\,dx\,dy$

27. $\int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} x^2y^2\,dx\,dy$

28. $\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\,dy\,dx$

29. A swimming pool is circular with a 40-ft diameter. The depth is constant along east-west lines and increases linearly from 2 ft at the south end to 7 ft at the north end. Find the volume of water in the pool.

30. An agricultural sprinkler distributes water in a circular pattern of radius 100 ft. It supplies water to a depth of e^{-r} feet per hour at a distance of r feet from the sprinkler.

(a) What is the total amount of water supplied per hour to the region inside the circle of radius R centered at the sprinkler?

(b) Determine an expression for the average amount of water per hour per square foot supplied to the region inside the circle of radius R.

31. Use polar coordinates to combine the sum

$$\int_{1/\sqrt{2}}^1 \int_{\sqrt{1-x^2}}^x xy\,dy\,dx + \int_1^{\sqrt{2}} \int_0^x xy\,dy\,dx + \int_{\sqrt{2}}^2 \int_0^{\sqrt{4-x^2}} xy\,dy\,dx$$

into one double integral. Then evaluate the double integral.

32. (a) We define the improper integral (over the entire plane $\mathbb{R}^2$)

$$I = \iint_{\mathbb{R}^2} e^{-(x^2+y^2)}\,dA = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dy\,dx$$
$$= \lim_{a\to\infty} \iint_{D_a} e^{-(x^2+y^2)}\,dA$$

where D_a is the disk with radius a and center the origin. Show that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dA = \pi$$

(b) An equivalent definition of the improper integral in part (a) is

$$\iint_{\mathbb{R}^2} e^{-(x^2+y^2)}\,dA = \lim_{a\to\infty} \iint_{S_a} e^{-(x^2+y^2)}\,dA$$

where S_a is the square with vertices $(\pm a, \pm a)$. Use this to show that

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx \int_{-\infty}^{\infty} e^{-y^2}\,dy = \pi$$

(c) Deduce that

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$$

(d) By making the change of variable $t = \sqrt{2}\,x$, show that

$$\int_{-\infty}^{\infty} e^{-x^2/2}\,dx = \sqrt{2\pi}$$

(This is a fundamental result for probability and statistics.)

33. Use the result of Exercise 32 part (c) to evaluate the following integrals.

(a) $\int_0^\infty x^2 e^{-x^2}\,dx$ (b) $\int_0^\infty \sqrt{x}\,e^{-x}\,dx$

12.5 Applications of Double Integrals

We have already seen one application of double integrals: computing volumes. Another geometric application is finding areas of surfaces and this will be done in the next section. In this section we explore physical applications such as computing mass, electric charge, center of mass, and moment of inertia. We will see that these physical ideas are also important when applied to probability density functions of two random variables.

Density and Mass

In Chapter 6 we were able to use single integrals to compute moments and the center of mass of a thin plate or lamina with constant density. But now, equipped with the double integral, we can consider a lamina with variable density. Suppose the lamina occupies a region D of the xy-plane and its **density** (in units of mass per unit area) at